

**Lyapunov exponents of the  
predator prey model**

**Ljapunovovy exponenty modelu  
dravec kořist**

# Diploma Thesis Assignment

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Title:

Ljapunovovy exponenty modelu dravec kořist  
Lyapunov exponents of the predator prey model

Description:

V inženýrských vědách se setkáváme s modely, které vykazují chaotické chování. Samotný pojem “chaos” je však definován různě, v závislosti na studiu příslušných vlastností. Klasickou definicí je “Devaney” chaos, který předpokládá hustotu periodických bodů a tranzitivitu. Mezi časté indikátory chaosu se také považují kladné “Ljapunovovy exponenty”.

Tématem práce je detekce Ljapunovových exponentů modelů typu dravec kořist.

Zásady pro vypracování práce:

1. seznámit se s pojmy,
2. provést analýzu Ljapunovových exponentů daného modelu,
3. konstruovat vhodný algoritmus k výpočtu Ljapunovových exponentů,
4. formulovat závěr, detekce chaosu z Ljapunovových exponentů.

Many phenomena arising from Engineering are chaotic. There are several definitions of “chaos” and it is defined by properties depending on the behavior. The classical definition was given by R. Devaney that is, the set of periodic points is dense and the model is transitive. Moreover, there are other indicators of chaos, i.e. positive Lyapunov exponents.

The main aim of the thesis is to detect Lyapunov exponents of the predator prey model.

The tools for the thesis:

1. study and define all needed notions,
2. analyze Lyapunov exponents of the model,
3. construct a suitable algorithm for the calculation of Lyapunov exponent,
4. formulate conclusions, how to detect chaos from the Lyapunov exponent.

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Extent and terms of a thesis are specified in directions for its elaboration that are opened to the public on the web sites of the faculty.

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I declare that I have worked up this thesis by myself. I have referenced all the sources and publications I have used.

Ostrava, May 3, 2013

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## Abstrakt

Cílem této práce je indikovat a kvantifikovat chaos v diskrétních dynamických systémech závislých na reálných parametrech. Nejprve je studován Ljapunovův exponent jedno-rozměrných zobrazení. Nejprve je podrobně zkoumán Logistický parametrický systém, jehož Ljapunovovy exponenty jsou počítány. Následně jsou uvedeny a studovány Ljapunovovy exponenty pro  $n$ -rozměrná zobrazení. Konkrétně je studováno Hénonovo zobrazení. Cílem zkoumání je vyšetření parametrického systému odvozeného od Lotka–Volterra zobrazení. K výpočtům a simulacím Ljapunovových exponentů byly užity algoritmy, uvedené v závěru práce.

**Klíčová slova:** Ljapunovův exponent, chaos, ergodicita, Logistické zobrazení, Hénonovo zobrazení, Lotka–Volterra

## Abstract

The main aim of this thesis is to indicate and quantify chaos in discrete dynamical systems that are depending on real parameters. As a first step the Lyapunov exponent is studied for one-dimensional maps. More precisely, the Logistic parametric family is deeply researched and the Lyapunov exponents are computed. Secondly, the Lyapunov exponents are introduced, studied and computed for  $n$ -dimensional maps, the Hénon map is investigated. As a goal of the research the Lotka–Volterra parametric family is introduced and utilizing installed algorithms their Lyapunov exponents are simulated and evaluated.

**Keywords:** Lyapunov exponent, chaos, ergodicity, Logistic map, Hénon map, Lotka–Volterra

## List of Used Abbreviations and Symbols

$C^n(X)$	– set of all $n$ times continuously differentiable maps on $X$
$f^i$	– $i$ -th iteration of the map $f$
$f _S$	– restriction of $f$ to $S$
$\mathbb{R}$	– set of all real numbers
$\mathbb{R}^n$	– set of all $n$ -dimensional real vectors

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# 1 Introduction

Dynamical systems are often used to describe phenomena in various fields of science like physics, chemistry, biology, economy etc. Many years scientists believed that behavior of systems given by differential or difference equations can be totally described as periodic or quasi-periodic. They believed that systems, which behavior is unpredictable, are just very complex. Later it was shown that such unpredictability can occur in simple systems and new phenomenon, chaos, was described. In this section we briefly show the most important events in the history of chaos theory.

From 1885 H. Poincaré studied the three-body problem. This problem describes the motion of bodies in simplified solar system [37]. The first and crucial development in the field of dynamical systems was made by H. Poincaré in 1890 [26] by *recurrence*, that is a point returns to itself arbitrarily close under the actions (iterations), or equivalently the point belongs to its omega limit set. Hence, a dynamical system preserves volume, all trajectories return arbitrarily close to their initial position and they do this an infinite number of times. More precisely, H. Poincaré discovered: *If a flow preserves volume and has only bounded orbits then for each open set there are orbits that intersect the set infinitely often.*

The Lyapunov stability, named after Aleksandr Lyapunov, describes the behavior of points in the neighborhood of the equilibria [23]. The equilibrium is Lyapunov stable if orbits of all the points in the neighborhood of that equilibrium don't leave the neighborhood. The equilibrium is asymptotically stable if orbits of all points that are in the neighborhood of that equilibrium converge to the equilibrium. Stronger notion is exponential stability which guarantees the minimal convergence speed. Later Lyapunov extended his studium of neighboring points to any two infinitesimally close points. The notion of Lyapunov exponent was set to measure the rate at which orbits of two infinitesimally close points diverge.

In the first half of the next century the ergodic theory studied dynamical systems motivated by some physical and mechanical problems. In 1920 G. D. Birkhoff provide the theorem which says that in dynamical systems given by continuous and transitive map there exists a set that has a dense orbit in the phase space. The notion of transitivity was introduced by G. D. Birkhoff in 1920 for flows [5]. The map  $F$  is transitive if for any two non-empty open sets  $U, V$ , that are subsets of the phase space, there is  $n \in \mathbb{N}$  such that  $F^n(U) \cap V \neq \emptyset$ .

But the most important idea in ergodic theory came in 1931. It was the Birkhoff Ergodic Theorem [6, Chapter 3] which says that for given integrable function  $f$  and measure preserving transformation  $T$  is the average of  $f$  along the orbit of  $T$  equal to the integral of  $f$ .

The foundation of the chaos theory, as it is known today, was in 1975 when T. Y. Li and J. A. Yorke set the definition of chaos [22]. The map is chaotic in sense of Li and Yorke if

there is an uncountably infinite set that is *scrambled* under that map. Roughly speaking it means that every pair of points in this set come infinitely close under some iteration of the map without staying close. T. Y. Li and J. A. Yorke proved that period three implies chaos for continuous self maps on the interval. This result was improved by J. Smítal in [32] where it is shown that a two point scrambled set yields uncountable one for continuous maps on the interval. The Sharkovskii's theorem says that if the continuous map on interval has periodic point of period three then it has periodic points of all periods, so-called Sharkovskii's ordering was built (see [15, page 44]). Let us point out that it is not possible to generalize Sharkovskii's theorem for general maps or spaces. It suffices to take rational rotation on the circle as counterexample.

With other definition of chaos came later R. L. Devaney [8] in 1989. The map is chaotic in the sense of Devaney if it has dense set of periodic points, it is transitive and sensitive on initial conditions. This definition became famous for the third assumption, the sensitive dependence on initial conditions. But later it was proved [4] that sensitive dependence on initial conditions is superfluous.

In [33] authors shew that the chaos in sense of Li-Yorke or Devaney is not stable. There exist functions that exhibit chaotic behavior but lose it under arbitrary small perturbations. The stronger version of Li-Yorke chaos, called the distributional chaos, was introduced [32]. The distributional chaos is defined by sequence of distribution functions of distances between the orbits of two points. The system is said to be deistributionally chaotic if there exists a pair of points for which the limes inferior of such sequence is not equal to its limes superior.

There exist other definitions of chaos, in [21] on  $\omega$ -chaos, in [36] on Devaney's chaos, in [16] on Block-Coppel's chaos, in [28] on Robinson's chaos, in [12] on Kato's chaos and in [3], [13] on mixing properties.. Relationships between different kinds of chaos were studied by many authors but there are still unsolved implications [14], [20]. That is why we can not determine which definition is the strongest.

Now when the chaotic behavior is defined we want to indicate such behavior and quantify it. The question is how to detect if the system is chaotic. And if it is that case, "how much" is the system chaotic? In this thesis we focus on one technique which is used for indication of chaotic behavior and which can quantify it, the Lyapunov exponent is explored.

## Organization of the thesis

In the next section, *Preliminaries*, the basic definitions and theorems are set, which are used in later parts. The most important part of this section is the Birkhoff Ergodic Theorem (Theorem 2.22) which is very useful tool for computing the Lyapunov exponents.

In the section *Lyapunov exponents for one-dimensional maps* the Lyapunov exponent is defined in dimension one and it is shown how it can be computed (directly in Example 3.7 or using The Birkhoff Ergodic Theorem in Example 3.10). We also show that two topologically conjugate systems have the same Lyapunov exponents (Theorem 3.9 and Example 3.10).

Lyapunov exponents of the Logistic family are studied in the section *Logistic family*. We show the dependence of the Lyapunov exponent on the real parameter and its relationship with the dynamical properties such as stability or dependence on initial conditions. The Logistic family is the classical example for the study of dynamical properties of discrete dynamical systems and chaos (see [8], [9], [24]). Moreover the definition of Devaney chaos (Definition 2.11) is set in this section. In the end of the section the bifurcation diagrams of Logistic family are shown and the period doubling route to chaos is observed.

In the first part of the section *Lyapunov exponents for  $n$ -dimensional maps* the analogy between one-dimensional case and multi-dimensional case is shown and the geometrical representation of the Lyapunov exponents are discussed. The way to approximate the Lyapunov exponents of  $n$ -dimensional maps is shown on 2-dimensional examples. In the section *Hénon map* we show the methods from previous parts on one of the best-known two-dimensional examples, the Hénon map. The numerical approximations of the Lyapunov exponents are computed for  $0 < a \leq 1.4$  and  $0 < b \leq 0.3$  since it changes behavior from ordered to chaos.

In the next section, *Lyapunov exponents of a Lotka–Volterra systems*, the Lotka–Volterra family and its competitive extension are introduced. Some special cases of competitive Lotka–Volterra, such as Sharkovskii’s system, are modified by real parameters and their behavior is studied as parameters change. The Lyapunov exponents are computed and their relationship with the bifurcation diagram is observed.

The programming language Python 2.7 in which all the numerical experiments were implemented is introduced in the last section. The implementations of numerical experiments are shown and the algorithms described.

## Goals of the thesis

The main aim of this thesis is to quantify chaos. The tool, used to do so, is the Lyapunov exponent which measures the divergence speed of two infinitesimally close points. This thesis focuses on the computing of the Lyapunov exponents of the one and multidimensional parametric families, namely the special cases of competitive Lotka–Volterra system. The algorithm for numerical approximation of the Lyapunov exponents is given.

## 2 Preliminaries

In this section we set the basic definitions, lemmas and theorems, which will be needed in later parts of the thesis. In section *Elementary dynamics* we recall some basic definitions and theorems to describe properties of discrete dynamical systems (see [7]). In section *Introduction to ergodicity* we briefly introduce some of the most important ideas of ergodic theory. For more consult [6].

### 2.1 Elementary dynamics

**Definition 2.1** Let  $X$  be a compact metric space and  $f : X \rightarrow X$  be a continuous map. By a discrete dynamical system is an ordered pair  $(X, f)$ .

The set  $X$  is called *phase space* or *state space* and the map  $f$  is called *evolution function*. In this thesis can be found the notation of systems given just by evolution function. In that case as the phase space is understood the strongly invariant set of the evolution function. The strongly invariant set is such set  $M$  for which holds  $f(M) = M$ .

**Definition 2.2** A point  $x$  is called the fixed point of the map  $f$  if  $f(x) = x$  and by  $\text{Fix}(f)$  we denote the set of all fixed points of the map  $f$ . A point  $x$  is called the periodic point of period  $n$  if  $f^n(x) = x$  and  $f^k(x) \neq x$  for  $0 < k < n$ . By  $\text{Per}_n(f)$  we denote the set of all periodic points of period  $n$  for map  $f$ .

By  $\text{Per}(f)$  we denote the set of all periodic points of  $f$ . By a cycle of period  $n$  we understand the sequence of points  $x_1, \dots, x_n \in \text{Per}_n(f)$  such that  $f(x_k) = x_{k+1}$  for  $k = 1, \dots, n-1$  and  $f(x_n) = x_1$ .

For the next definition suppose that Jacobian matrix of  $f^n$  at  $x$  exists.

**Definition 2.3** Periodic point  $x \in \text{Per}_n(f)$  is called hyperbolic if the Jacobian matrix  $J_{f^n}(x)$  has no eigenvalues on the unit circle.

**Definition 2.4** Let  $x$  be a hyperbolic periodic point.

- (i)  $x$  is called sink or attracting periodic point if the absolute values of all the eigenvalues of  $J_{f^n}$  are less than one.
- (ii)  $x$  is called source or repelling periodic point if the absolute values of all the eigenvalues of  $J_{f^n}$  are greater than one.
- (iii)  $x$  is called saddle point if some of the absolute values of the eigenvalues of  $J_{f^n}$  are less than one and some are greater than one.

#### Example 2.5

- Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^3$ . Then  $\text{Fix}(f) = \{0, 1\}$ . Point  $x = 0$  is attracting fixed point and  $x = 1$  is repelling fixed point.

- Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $f(x, y) = (x(4 - x - y), xy)$ . Then  $\text{Fix}(f) = \{(0, 0), (1, 2), (3, 0)\}$ . Point  $(x, y) = (0, 0)$  is saddle point and the other points are sources.

Now let us introduce the topological conjugacy. It is a strong tool which simplifies detecting some dynamical properties such as chaos. In Theorem 3.9 we prove that two topologically conjugate systems have the same Lyapunov exponents.

**Definition 2.6** Let  $X, Y$  be topological spaces and  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be continuous maps. We say that  $f$  is topologically conjugate to  $g$  if there exists homeomorphism  $\phi : X \rightarrow Y$  such that  $\phi \circ f = g \circ \phi$ . Such homeomorphism is called the conjugacy.

Topological conjugacy is the equivalence relation since it is reflexive (conjugacy map is the identity), symmetric (if  $f$  and  $g$  are conjugate by  $h$  then  $g$  and  $f$  are conjugate by  $h^{-1}$ ) and transitive (if  $f$  is conjugate to  $g_1$  by  $h_1$  and  $g_1$  to  $g_2$  by  $h_2$  then  $f$  is conjugate to  $g_2$  by  $h_1 \circ h_2$ ).

**Definition 2.7** A family of dynamical systems is called topologically equivalent if it is closed under topological conjugacy, that is there exists a conjugacy between every two systems.

**Proposition 2.8** Let  $f$  and  $g$  be topologically conjugate by conjugacy  $h$  and let  $x \in \text{Per}_n(f)$ . Then  $h(x) \in \text{Per}_n(g)$ .

### Example 2.9

The fixed points of the Tent map are  $\text{Fix}(T) = \{0, 2/3\}$ . Tent map is topologically conjugate to the Logistic map by conjugacy  $h(x) = \sin^2(\pi x/2)$ , hence the fixed points of the Logistic map are  $\text{Fix}(F_4) = \{h(0), h(2/3)\} = \{0, 3/4\}$ .

In this thesis is chaos defined in Devaney's sense since the sensitive dependence on initial conditions is assumed. Later we use that assumption to indicate the chaotic behavior. To define Devaney's chaos we need to define transitivity first.

**Definition 2.10** Transformation  $f : X \rightarrow X$  is said to be topologically transitive if for any pair of open sets  $U, V \subset X$  there exists  $k > 0$  such that  $f^k(U) \cap V \neq \emptyset$ .

**Definition 2.11** [8, page 50] Let  $X$  be a set. The map  $f : X \rightarrow X$  is said to be chaotic in  $X$  if

1.  $f$  has sensitive dependence on the initial conditions.
2.  $f$  is topologically transitive.
3. periodic points are dense in  $X$ .

**Remark 2.12** In [4] it is shown that the first assumption is superfluous. The following statement was proved for continuous maps on the compact metric spaces.

"If  $f : X \rightarrow X$  is transitive and has dense periodic points then  $f$  has sensitive dependence on initial conditions."

Let us note that topological transitivity and chaos in sense of Devaney are preserved under topological conjugacy.

## 2.2 Introduction to ergodicity

In this section we set the basic definitions and theorems from ergodic theory and introduce the Birkhoff Ergodic Theorem which is a strong tool used in later sections.

**Definition 2.13** Let  $(X_1, \mu_1)$  and  $(X_2, \mu_2)$  be measure spaces. We say that mapping  $f : X_1 \rightarrow X_2$  is measurable if  $f^{-1}(E)$  is measurable for every measurable subset  $E \subset X_2$ . The mapping is measure preserving if  $\mu_1(f^{-1}(E)) = \mu_2(E)$  for every measurable subset  $E \subset X_2$ . When  $X_1 = X_2$  and  $\mu_1 = \mu_2$ , we call  $f$  a transformation.

If a measurable transformation  $f : X \rightarrow X$  preserves a measure  $\mu$ , then we say that  $\mu$  is  $f$ -invariant (or invariant under  $f$ ). If  $f$  is invertible and if both  $f$  and  $f^{-1}$  are measurable and measure preserving, then we call  $f$  an invertible measure preserving transformation.

### Example 2.14

- The identity map is obviously invertible and Lebesgue measure preserving transformation on  $[0, 1]$ .
- The Tent map defined by

$$T(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}), \\ 2 - 2x & x \in [\frac{1}{2}, 1], \end{cases}$$

is Lebesgue measure preserving transformation. Since Tent is measurable two-to-one map up to the point  $x = 1$ , we can for every measurable set  $E$  split the set  $E^{-1} = T^{-1}(E)$  on two sets  $E_1^{-1}, E_2^{-1}$ , one in  $[0, 1/2]$  and the other in  $[1/2, 1]$ . Each of these sets has measure  $\lambda(E_1^{-1}) = \lambda(E_2^{-1}) = \lambda(E)/2$ . Hence,  $\lambda(E^{-1}) = \lambda(E_1^{-1} \cup E_2^{-1}) = \lambda(E_1^{-1}) + \lambda(E_2^{-1}) = \lambda(E)$ , showing that Tent map preserves Lebesgue measure. Tent map is not invertible on  $[0, 1]$  (see Figure 1).

- The transformation, given by

$$f(x) = \begin{cases} 2x & x \in [0, \frac{1}{4}), \\ \frac{1}{2} & x \in [\frac{1}{4}, \frac{3}{4}), \\ 2x - 1 & x \in [\frac{3}{4}, 1], \end{cases}$$

is obviously a measurable transformation, but it doesn't preserve the Lebesgue measure.

- The Logistic map defined by

$$F_4(x) = 4x(1 - x)$$

doesn't preserve Lebesgue measure on  $[0, 1]$ .



Let  $E = [0.4, 0.6]$ . Then

$$\begin{aligned}\lambda(E) &= 0.2 \quad \text{and} \\ \lambda(f^{-1}(E)) &= \lambda([a, b] \cup [c, d]) = \\ &= \lambda([a, b]) + \lambda([c, d]) \approx 0.142141,\end{aligned}$$

where  $a = f^{-1}|_{[0, 1/2]}(0.4)$ ,  $b = f^{-1}|_{[0, 1/2]}(0.6)$ ,  $c = f^{-1}|_{[1/2, 1]}(0.6)$  and  $d = f^{-1}|_{[1/2, 1]}(0.4)$ . Consequently

$$\lambda(E) \neq \lambda(f^{-1}(E)).$$

But for measure defined by

$$\mu(E) = \int_E \frac{1}{\pi \sqrt{x(1-x)}} dx$$

it is measure preserving on  $[0, 1]$  (see [6]).

**Lemma 2.15 (Fatou's lemma)** Let  $f_n \geq 0, n \in \mathbb{N}$  be a sequence of measurable functions on a measure space  $(X, \mu)$ . Then

$$\int_X \liminf f_n d\lambda \leq \liminf \int_X f_n d\lambda.$$

Let us recall the notion of *probability space*. The measure space  $(X, \mathcal{A}, \mu)$  is called probability space if  $\mu(X) = 1$ .

**Definition 2.16** Let  $(X, \mathcal{A}, \mu)$  be a probability space. Suppose that  $f : X \rightarrow X$  is  $\mu$ -invariant. Then  $f$  is said to be *ergodic* if  $E \in \mathcal{A}$  satisfies  $f^{-1}(E) = E$  if and only if  $\mu(E) = 0$  or  $1$ .

If there exist pairwise disjoint measurable subsets  $E_j$  satisfying  $\mu(E_j) > 0$ ,  $X = \bigcup_j E_j$ ,  $f(E_j) \subset E_j$ , and if  $f|_{E_j} : E_j \rightarrow E_j$  is ergodic with respect to the conditional measure  $\mu_{E_j}$ , then each  $E_j$  is called an *ergodic component* of  $f$ .

**Theorem 2.17** The following statements are equivalent:

- (i)  $f$  is ergodic.
- (ii) If  $\mu(A) > 0$  then  $\bigcup_{n=1}^{\infty} f^{-n}(A) = X$ .
- (iii) If  $\mu(A) > 0$  and  $\mu(B) > 0$ , then  $\mu(f^{-n}(A) \cap B) > 0$  for some  $n \geq 1$ .
- (iv) If a measurable function  $g$  satisfies  $g(f(x)) = g(x)$  for almost every  $x$ , then  $g$  is constant almost everywhere.

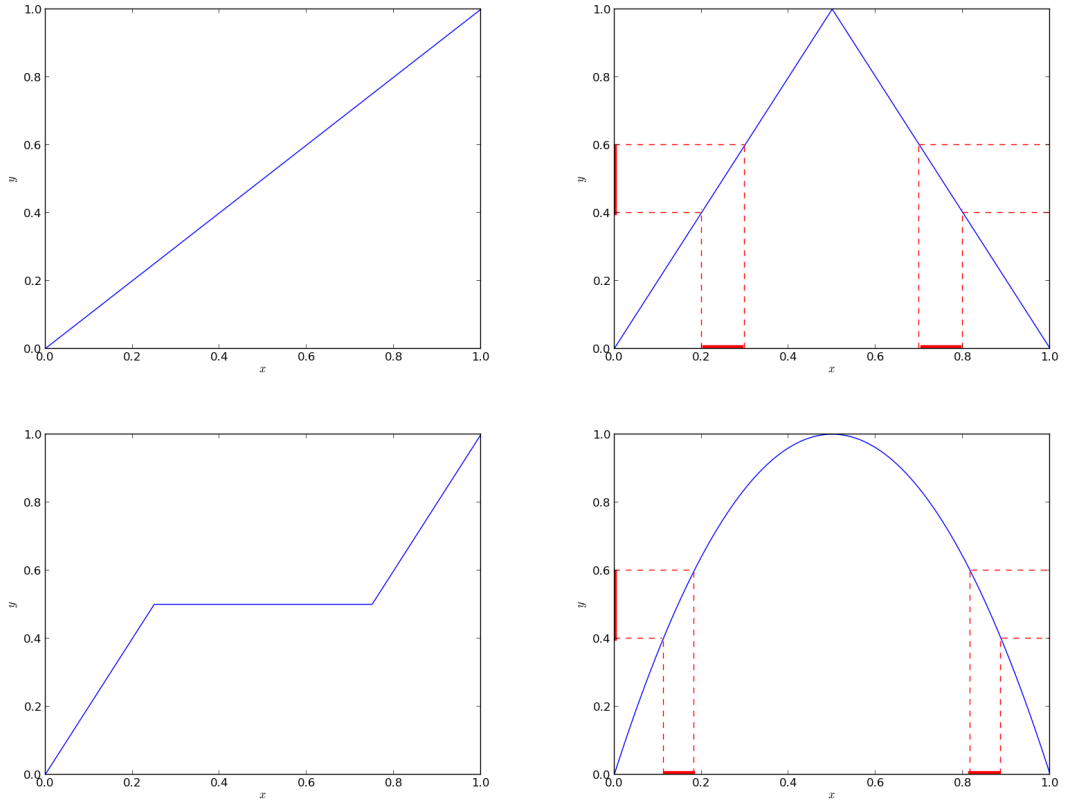


Figure 1: Identity, Tent map, measurable transformation and Logistic map

**Proof.** (i)  $\Rightarrow$  (ii). Put  $E = \bigcup_{n=1}^{\infty} f^{-n}(A)$ . Then

$$f^{-1}(E) = \bigcup_{n=2}^{\infty} f^{-n}(A) \subset E$$

and  $\mu(E \Delta f^{-1}(E)) = \mu(E) - \mu(f^{-1}(E)) = 0$ . Hence  $E = f^{-1}(E)$ . Since  $f$  is ergodic,  $E = \emptyset$  or  $E = X$ . Since  $E$  contains  $A$ , we conclude that  $E = X$ .

(ii)  $\Rightarrow$  (iii). Since  $B = \bigcup_{n=1}^{\infty} (f^{-n}(A) \cap B)$ , there exists  $n \geq 1$  such that  $\mu(f^{-n}(A) \cap B) > 0$ .

(iii)  $\Rightarrow$  (i). Suppose  $B$  satisfies  $f^{-1}(B) = B$  and  $\mu(B) > 0$ . Choose  $A = X \setminus B$ . Then  $f^{-n}(A) = X \setminus f^{-n}(B) = X \setminus B$ , and hence  $\mu(f^{-n}(A) \cap B) = 0$  for any  $n \geq 1$ . Thus  $\mu(A) = 0$  and  $\mu(B) = 1$ .

(i)  $\Rightarrow$  (iv). Let  $g$  be a measurable function such that  $g(f(x)) = g(x)$  almost everywhere. By considering real and imaginary parts, we may assume that  $g$  is real-valued. Put

$$E_{n,k} = \{x \in X : 2^{-n}k \leq g(x) < 2^{-n}(k+1)\}$$

for  $n \geq 1$  and  $k \in \mathbb{Z}$ . Then  $\{E_{n,k} : k \in \mathbb{Z}\}$  is a partition of  $X$  for every  $n$ .

Note that

$$\begin{aligned} f^{-1}(E_{n,k}) &= \{x \in X : 2^{-n}k \leq g(Tx) < 2^{-n}(k+1)\} \\ &= \{x \in X : 2^{-n}k \leq g(x) < 2^{-n}(k+1)\} \\ &= E_{n,k} \end{aligned}$$

since  $g(f(x)) = g(x)$ . Hence  $E_{n,k}$  has measure 0 or 1; more precisely, for each  $n$  there exists a unique value for  $k$ , say  $k_n$ , such that  $E_{n,k_n}$  has measure 1 and  $E_{n,k}$  has measure 0 for  $k \neq k_n$ . Let  $X_0 = \bigcap_{n=1}^{\infty} E_{n,k_n}$ . Then  $\mu(X_0) = 1$  and  $g$  is constant on  $X_0$ .

(iv)  $\Rightarrow$  (i). Suppose that  $f^{-1}(E) = E$ . Then  $g(x) = 1_E(x)$  satisfies  $g(f(x)) = g(x)$ , and hence  $g = 1_E$  is constant. Since the possible values of  $1_E$  are 0 and 1, we conclude that either  $1_E = 0$  almost everywhere or  $1_E = 1$  almost everywhere, which is equivalent to  $\mu(E) = 0$  or  $\mu(E) = 1$ . ■

### Example 2.18 (Translations modulo 1)

Let  $X = [0, 1)$  and  $f(x) = x + \theta \pmod{1}$ . Then  $f$  is ergodic with respect to Lebesgue measure if and only if  $\theta$  is irrational.

Suppose that  $\theta$  is rational number of a form  $\theta = p/q$ , where  $p, q \in \mathbb{N}$ . Then  $e^{2\pi i q x}$  is a nonconstant  $f$ -invariant function. Hence  $f$  is not ergodic.

If  $\theta$  is irrational and  $g(x) = \sum_{n=0}^{\infty} c_n e^{2\pi i n x}$  is the Fourier series expansion of an invariant function  $g$ , then

$$g(x + \theta) = \sum_{n=0}^{\infty} c_n e^{2\pi i n \theta} e^{2\pi i n x}$$

and  $c_n = c_n e^{2\pi i n \theta}$  for every  $n$ . If  $c_n \neq 0$  for some  $n \neq 0$  then  $1 = e^{2\pi i n \theta}$ , which is impossible since  $\theta$  is irrational. Hence  $g = c_0$ .

Other examples of ergodic maps (e.g. multiplication by 2 modulo 1, two sided shift, toral endomorphisms etc.) can be found in [6] and [35].

Returning to the technique of topological conjugacy we can observe following.

### Example 2.19

The Tent map preserves Lebesgue measure on  $[0, 1]$  and is topologically conjugate to the Logistic map (for the conjugacy function see Example 3.10) which doesn't preserve Lebesgue measure. Moreover the conjugacy function doesn't have to be measure-preserving. Logistic map is topologically conjugate to the map  $g(x) = x(4 - x)$  by conjugacy  $h(x) = 4x$  which doesn't preserve measure. Hence, the topological conjugacy doesn't preserve measure.

Before we introduce the Birkhoff Ergodic Theorem we need the following lemma and corollary to prove the theorem. Both proofs can be found in [6, Chapter 3] but are given for completeness.

**Lemma 2.20 (The Maximal Ergodic Theorem)** Let  $f : X \rightarrow X$  be a measure preserving transformation on a space  $(X, \mu)$ . (We do not exclude case that  $\mu(X) = \infty$ .) Let  $g$  be a real-valued integrable function on  $X$ .

Define  $g_0 = 0$ ,

$$g_n = g + g \circ f + \cdots + g \circ f_{n-1}$$

for  $n > 1$ , and

$$G_N = \max_{0 \leq n \leq N} g_n.$$

Put  $A_N = \{x : G_N(x) > 0\}$ . Then

$$\int_{A_N} g \, d\mu \geq 0.$$

**Proof.** Note that  $G_n \geq 0$  and  $G_n \in L^1(X, \mu)$ . For  $0 \leq n \leq N$  we have  $G_n \geq g_n$ , and so  $G_N \circ f \geq g_n \circ f$ . Hence  $G_N \circ f + g_n \geq g_{n+1}$ . Thus

$$G_N(f(x)) + g_n(x) \geq \max_{1 \leq n \leq N} g_n(x).$$

If  $G_n(x) > 0$ , then the right side is equal to  $\max_{0 \leq n \leq N} g_n(x) = G_N(x)$ , and hence  $g \geq G_N - G_N \circ f$  on  $A_N$ . Now we have

$$\int_{A_N} g \geq \int_{A_N} G_N - \int_{A_N} G_N \circ f = \int_X G_N - \int_{A_N} G_N \circ f$$

since  $G_N = 0$  on  $X \setminus A_N$ . Since  $G_N \circ f \geq 0$ , we conclude that

$$\int_{A_N} g \geq \int_X G_N - \int_{A_N} G_N \circ f = 0,$$

where the last equality comes from the  $\mu$ -invariance of  $f$ . ■

**Corollary 2.21** Let  $f : X \rightarrow X$  be measure preserving on a finite measure space  $(X, \mu)$ . If  $h : X \rightarrow \mathbb{R}$  is integrable and

$$B_\alpha = \left\{ x \in X : \sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} h(f^i(x)) > \alpha \right\},$$

then

$$\int_{B_\alpha} h \, d\mu \geq \alpha \mu(B_\alpha).$$

If  $f^{-1}(A) = A$ , then

$$\int_{B_\alpha} h \, d\mu \geq \alpha \mu(A \cap B_\alpha).$$

**Proof.** Let  $g = h - \alpha$ . Then  $B_\alpha = \bigcup_{N=0}^{\infty} \{x : G_N(x) > 0\}$  and  $\int_{B_\alpha} g \, d\mu > 0$  by the Maximal Ergodic Theorem. Hence

$$\int_{B_\alpha} h \, d\mu - \alpha \mu(B_\alpha) \geq 0.$$

For the second part we consider the restriction of  $f$  to  $A$ . In this case the invariant subset  $A$  plays the role of  $X$  in the first case. ■

**Theorem 2.22 (The Birkhoff Ergodic Theorem)** *Let  $(X, \mu)$  be a probability space. If  $f$  is  $\mu$ -invariant and  $g$  is integrable, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(x)) = g^*(x)$$

for some  $g^* \in L^1(X, \mu)$  with  $g^*(f(x)) = g^*(x)$  for almost every  $x$ . Furthermore, if  $f$  is ergodic, then  $g^*$  is constant and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(x)) = \int_X g \, d\mu$$

for almost every  $x$ .

**Proof.** By considering real and imaginary parts we may prove the statement for real-valued functions  $g$ . Put

$$g^*(x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x))$$

and

$$g_*(x) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)).$$

It is clear that  $g^*(f(x)) = g^*(x)$  and  $g_*(f(x)) = g_*(x)$ . It remains to show that  $g^* = g_*$  and that they are integrable.

For rational numbers  $\alpha, \beta$  put

$$E_{\alpha, \beta} = \{x \in X : g_*(x) < \beta \text{ and } \alpha < g^*(x)\}.$$

Note that  $\{x \in X : g_*(x) < g^*(x)\} = \bigcup_{\beta < \alpha} E_{\alpha, \beta}$  (where  $\alpha, \beta$  are both rational) and  $f^{-1}(E_{\alpha, \beta}) = E_{\alpha, \beta}$ . Put

$$B_\alpha = \left\{ x \in X : \sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) > \alpha \right\}.$$

Then  $E_{\alpha, \beta} \subset B_\alpha$ . From the above corollary we have

$$\int_{E_{\alpha, \beta}} g \, d\mu = \int_{E_{\alpha, \beta} \cap B_\alpha} g \, d\mu \geq \alpha \mu(E_{\alpha, \beta} \cap B_\alpha) = \alpha \mu(E_{\alpha, \beta}),$$

and hence  $\int_{E_{\alpha,\beta}} g \, d\mu \geq \alpha \mu(E_{\alpha,\beta})$ .

Note that  $(-g)^* = -g_*$ ,  $(-g)_* = -g^*$  and

$$E_{\alpha,\beta} = \{x : (-g)^*(x) > -\beta \text{ and } -\alpha > (-g)_*(x)\}.$$

If we replace  $g, \alpha, \beta$  by  $-g, -\beta, -\alpha$ , respectively in the previous inequality, then we have  $\int_{E_{\alpha,\beta}} (-g) \, d\mu \geq -\beta \mu(E_{\alpha,\beta})$ , and hence  $\int_{E_{\alpha,\beta}} g \, d\mu \leq \beta \mu(E_{\alpha,\beta})$ .

Thus we obtain  $\alpha \mu(E_{\alpha,\beta}) \leq \beta \mu(E_{\alpha,\beta})$ , which implies that if  $\beta < \alpha$  then  $\mu(E_{\alpha,\beta}) = 0$ . Therefore  $g^* = g_*$  and  $\frac{1}{n} \sum_{i=1}^{n-1} g(f^i(x))$  converges to  $g^*$  almost everywhere.

Now we show that  $g^*$  is integrable. Let

$$h_n(x) = \left| \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) \right|.$$

Then  $\lim h_n(x) = |g^*|$  and

$$\int h_n \, d\mu \leq \frac{1}{n} \sum_{i=0}^{n-1} \int |g(f^i(x))| \, d\mu = \int |g| \, d\mu.$$

Fatou's lemma implies that

$$\int |g^*| \, d\mu = \int \liminf h_n \, d\mu = \liminf \int h_n \, d\mu \leq \int |g| \, d\mu < \infty.$$

It remains to show that  $\int g \, d\mu = \int g^* \, d\mu$ . For  $n \geq 1$  and  $k \in \mathbb{Z}$  put

$$D_{n,k} = \left\{ x \in X : \frac{k}{n} \leq g^*(x) < \frac{k+1}{n} \right\}.$$

Note that for each  $n$  the collection of  $D_{n,k}$ ,  $k \in \mathbb{Z}$ , forms a partition of  $X$ . For sufficiently small  $\varepsilon > 0$  we have

$$D_{n,k} \subset B_{\frac{k}{n} - \varepsilon}.$$

Hence by Corollary 2.21 we have

$$\int_{D_{n,k}} g \, d\mu \geq \left( \frac{k}{n} - \varepsilon \right) \mu(D_{n,k})$$

for sufficiently small  $\varepsilon > 0$ , and hence

$$\int_{D_{n,k}} g \, d\mu \geq \frac{k}{n} \mu(D_{n,k}).$$

By the definition of  $D_{n,k}$

$$\int_{D_{n,k}} g^* \, d\mu \leq \frac{k+1}{n} \mu(D_{n,k}) \leq \frac{1}{n} \mu(D_{n,k}) + \int_{D_{n,k}} g \, d\mu.$$

Summing over  $k$  we obtain

$$\int_X g^* d\mu \leq \frac{1}{n} + \int_X g d\mu.$$

This holds true for every  $n$ . By letting  $n \rightarrow \infty$  we have

$$\int_X g^* d\mu \leq \int_X g d\mu.$$

Applying the same procedure to  $-g$  we obtain

$$\int (-g)^* d\mu \leq \int (-g) d\mu,$$

and so

$$-\int g_* d\mu \leq -\int g d\mu.$$

Since  $g^* = g_*$ , we conclude that  $\int g d\mu = \int g^* d\mu$ . ■

The Birkhoff Ergodic Theorem says that the time average of integrable function  $f$  over ergodic transformation  $T$ , defined by

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \quad (\text{if the limit exists})$$

is equal to its space average, defined by

$$\int_X f(x) d\mu.$$

Motivation for the Birkhoff Ergodic Theorem could be the question with what frequency does the orbit of some point enter the subset  $E$ ? The Birkhoff Ergodic Theorem gives us the answer for measure preserving transformation  $T$ . If the transformation  $T$  is ergodic, the orbit of almost every point enters the set  $E \in X$  with asymptotic relative frequency  $\mu(E)$ .

The theorem is often used in number theory for study of frequencies of digits in sequences. The proofs of the *Borel Theorem on Normal Numbers* and  *$L^P$  Ergodic Theorem of Von Neumann* in number theory are based on the Birkhoff Ergodic Theorem (see [35]). The alternative definition of ergodicity can be set as the corollary of the Birkhoff Ergodic Theorem.

### 3 Lyapunov exponents for one-dimensional maps

In this section Lyapunov exponents are computed for one-dimensional maps. We introduce a numerical approximation of Lyapunov exponent and compute Lyapunov exponents of well-known maps such as Tent map or Logistic map. Here we show also what the Lyapunov exponent describes and how it depends on the behavior of the dynamical system.

**Definition 3.1** Let  $X = [a, b]$ ,  $a, b \in \mathbb{R}$  be an interval,  $\mu$  is an absolutely continuous invariant measure on that interval and  $f$  is a piecewise continuously differentiable map on  $X$ . The number

$$L(f) = \int_a^b \log |f'| \, d\mu$$

is called the Lyapunov exponent of  $f$ .

The alternative definition of Lyapunov exponent is

$$L(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left| f' \left( f^k(x) \right) \right| \quad a.e.$$

which can be obtained from Definition 3.1 by using the Birkhoff Ergodic Theorem (Theorem 2.22). This definition is often used since computing the integral in the Definition 3.1 can be hard.

The Lyapunov exponent is often used for indication of sensitive dependence on initial conditions. The system with negative Lyapunov exponent is considered stable and the system with positive Lyapunov exponent is considered unstable or chaotic. This isn't true for general maps. In [18] was proved that positive Lyapunov exponent implies sensitive dependence on initial conditions under some additional conditions for continuous maps on the closed interval (see Proposition 3.3). Moreover in [18] it was shown that the map is Lyapunov stable if the Lyapunov exponent is negative taking into account continuously differentiable maps on the closed interval (see Proposition 3.2).

**Proposition 3.2** Suppose  $f : [0, 1] \rightarrow [0, 1]$  is  $C^2$ . If the Lyapunov exponent  $L(f) < 0$  for some  $x_0$ , then the orbit of  $x_0$  is Lyapunov stable.

**Proposition 3.3** Suppose  $f : [0, 1] \rightarrow [0, 1]$  is  $C^2$ . If the Lyapunov exponent  $L(f) > 0$  for some  $x_0$  and the orbit of  $x_0$  satisfies

$$\inf_{n \geq 0} |f'(x_n)| > 0,$$

then the orbit exhibits sensitive dependence on initial conditions.

**Open question 3.4** (i) If a continuous map on interval has positive Lyapunov exponent, does it imply chaos? If it does, in which sense?



- (ii) If a continuous map on interval has Lyapunov exponent equal to zero, does it imply that there is a bifurcation boundary?
- (iii) If a continuous map on interval has negative Lyapunov exponent, does it imply Lyapunov stability?

The third point is partially solved by Proposition 3.2 which gives us the additional conditions under which it holds true.

**Definition 3.5** Let  $\mu$  be a measure on  $X \subset \mathbb{R}^n$ . If a measurable function  $\rho(x) \geq 0$  satisfies  $\mu(E) = \int_E \rho(x) dx$  for any measurable subset  $E$ , then  $\mu$  is said to be absolutely continuous and  $\rho$  is called a density function.

The following technical lemma will be used in the sequel.

**Lemma 3.6 [6]** Let  $\mu$  be an absolutely continuous measure on  $X \subset \mathbb{R}^n$  and  $\rho$  the density function. Then for every integrable function  $f$  the following statement holds

$$\int_X f(x) d\mu = \int_X f(x) \rho(x) dx.$$

**Example 3.7 (Logistic map)**

Let  $F_4$  be a logistic map, defined on  $X = [0, 1]$  by

$$F_4(x) = 4x(1-x).$$

The Lyapunov exponent of  $F_4$  can be calculated directly from the definition:

$$L(F_4(x)) = \int_X \log |F_4'(x)| d\mu.$$

For Logistic transformation the invariant probability density function is known (see [6]). It is

$$\rho(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

From Lemma 3.6 we get

$$\begin{aligned} L(F_4) &= \int_X \log |F_4'| \rho(x) dx = \int_0^1 \frac{\log |4-8x|}{\pi \sqrt{x(1-x)}} dx = \frac{1}{\pi} \int_0^1 \frac{\log 4 |1-2x|}{\sqrt{x(1-x)}} dx = \\ &= \frac{1}{\pi} \int_0^1 \frac{\log |1-2x|}{\sqrt{x(1-x)}} dx + \frac{1}{\pi} \int_0^1 \frac{2 \log 2}{\sqrt{x(1-x)}} dx \end{aligned}$$

We compute the second integral first. The expression under the square root can be completed to the square,

$$\frac{1}{\pi} \int_0^1 \frac{2 \log 2}{\sqrt{x(1-x)}} dx = \frac{2 \log 2}{\pi} \int_0^1 \frac{1}{\sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}} dx = \frac{2 \log 2}{\pi} \int_0^1 \frac{2}{\sqrt{1 - (2x - 1)^2}} dx.$$

We use substitution  $t = 2x - 1$  and get the result

$$\frac{2 \log 2}{\pi} \int_0^1 \frac{2}{\sqrt{1 - (2x - 1)^2}} dx = \frac{2 \log 2}{\pi} [\arcsin(2x - 1)]_0^1 = 2 \log 2$$

Since both  $\log |1 - 2x|$  and  $\sqrt{x(1 - x)}$  are symmetric with respect to  $x = 1/2$ , we can write

$$L(F_4(x)) = \frac{2}{\pi} \int_0^{1/2} \frac{\log |1 - 2x|}{\sqrt{x(1 - x)}} dx + 2 \log 2.$$

Now the task to solve is the following integral

$$\int_0^{1/2} \frac{\log |1 - 2x|}{\sqrt{x(1 - x)}} dx.$$

The expression under the square root can be completed to the square,

$$\int_0^{1/2} \frac{\log |1 - 2x|}{\sqrt{x(1 - x)}} dx = \int_0^{1/2} \frac{\log |1 - 2x|}{\sqrt{-\frac{1}{4}(1 - 2x)^2 + \frac{1}{4}}} dx.$$

Put  $1 - 2x = \sin \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = -\frac{1}{2} \cos \theta d\theta$  and

$$\int_0^{1/2} \frac{\log |1 - 2x|}{\frac{1}{2} \sqrt{1 - (1 - 2x)^2}} dx = \int_0^{\pi/2} \log \sin \theta d\theta = -\frac{\pi}{2} \log 2.$$

Hence the Lyapunov exponent of the Logistic map is

$$L(F_4(x)) = \log 2.$$

The second equality is proved in [6, page 273, Lemma 9.10.]. The lemma with proof is given for completeness.

**Lemma 3.8**

$$\int_0^{\pi/2} \log \sin \theta d\theta = -\frac{\pi}{2} \log 2.$$

**Proof.** Consider an analytic function  $f(z) = \log(1 - z)$ ,  $|z| \leq r < 1$ . Its real part  $u(z) = \log |1 - z|$  is harmonic, and hence the Mean Value Theorem for harmonic functions implies that

$$0 = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |1 - re^{i\theta}| d\theta.$$

Letting  $r \rightarrow 1$ , we have  $0 = \log |1 - e^{i\theta}| d\theta$ . Since  $|1 - e^{i\theta}| = 2 \sin \frac{\theta}{2}$ ,  $0 \leq \theta \leq 2\pi$ , we have

$$-2\pi \log 2 = \int_0^{2\pi} \log \sin \frac{\theta}{2} d\theta.$$

Substituting  $t = \theta/2$ , we obtain

$$-2\pi \log 2 = \int_0^\pi \log \sin t \, dt = 2 \int_0^{\pi/2} \log \sin t \, dt.$$

■

In the next theorem we show that two topologically conjugate systems have the same Lyapunov exponent.

**Theorem 3.9** *Let  $f$  and  $g$  be continuously differentiable transformations on  $[0, 1]$  preserving probability measures  $\rho_1(x) \, dx$  and  $\rho_2(x) \, dx$ , respectively. Suppose that there exists a continuous and piecewise differentiable conjugacy mapping  $\phi : [0, 1] \rightarrow [0, 1]$  such that*

$$g \circ \phi = \phi \circ f.$$

*We assume that  $f, g$  and  $\phi$  satisfy some integrability conditions as needed in the following proof. Then  $f$  and  $g$  have the same Lyapunov exponent.*

**Proof.** Since  $\phi$  is monotone, we assume that  $\phi(0) = 0$  and  $\phi(1) = 1$ . The other case is similar. Since  $\phi$  is measure preserving,

$$\int_0^x \rho_1(t) \, dt = \int_0^{\phi(x)} \rho_2(t) \, dt.$$

Hence  $\rho_1(x) = \rho_2(\phi(x)) \phi'(x)$ . From

$$g'(\phi(x)) \phi'(x) = \phi'(f(x)) f'(x),$$

we have

$$\begin{aligned} & \int_0^1 \log |g'(\phi(x))| \rho_1(x) \, dx + \int_0^1 \log |\phi'(x)| \rho_1(x) \, dx = \\ &= \int_0^1 \log |\phi'(f(x))| \rho_1(x) \, dx + \int_0^1 \log |f'(x)| \rho_1(x) \, dx. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^1 \log |g'(\phi(x))| \rho_1(x) \, dx &= \int_0^1 \log |g'(\phi(x))| \rho_2(\phi(x)) \phi'(x) \, dx = \\ &= \int_0^1 \log |g'(y)| \rho_2(y) \, dy, \end{aligned}$$

which is equal to the Lyapunov exponent of  $g$ . By the invariance of  $\rho_1 \, dx$  under  $f$ ,

$$\int_0^1 \log |\phi' \circ f| \rho_1 \, dx = \int_0^1 \log |\phi'| \rho_1 \, dx.$$

■

The following example uses the alternative definition to compute the Lyapunov exponent of the Tent map.

**Example 3.10 (Tent map)**

Let  $T$  be a tent map, defined by

$$T(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] , \\ -2x + 2 & x \in (\frac{1}{2}, 1] \end{cases}$$

The Lyapunov exponent of  $T$  can be computed as

$$L(T) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |T'(T^k(x))|.$$

for some initial  $x$ .

Since  $|T'(x)| = 2$  for any  $x \in (0, 1/2) \cup (1/2, 1)$  it is  $\log |T'(T^i(x))| = \log 2$ . Hence the Lyapunov exponent of  $T$  is equal to the mean value of  $\log 2$ .

$$L(T) = \log 2.$$

This is in agreement with Theorem 3.9 because the Tent map is topologically conjugate to the Logistic map by conjugacy  $h(x) = \sin^2(\frac{\pi}{2}x)$  which is piecewise differentiable on  $[0, 1]$ .

The geometrical representation of Lyapunov exponent is the divergence speed of two neighboring points. The following definition gives us the way to numerically approximate the value of the Lyapunov exponent by computing the distances of orbits of two close points.

**Definition 3.11** [6, page 276] Let  $x \in [a, b - 10^{-k}]$ ,  $a, b \in \mathbb{R}$  and  $\tilde{x} = x + 10^{-k}$ . Define

$$H_{n,k}(x) = \frac{1}{n} \log |f^n(x) - f^n(\tilde{x})| + \frac{k}{n}.$$

**Theorem 3.12** Suppose  $f$  is an ergodic and piecewise continuously differentiable map on  $X = [a, b]$ . For large enough  $k, n$  is  $H_{n,k}(x)$  close to the Lyapunov exponent of  $f$ .

**Proof.** Let  $x \in [a, b - 10^{-k}]$ ,  $a, b \in \mathbb{R}$  and  $\tilde{x} = x + 10^{-k}$  as in Definition 3.11

The following inequality is equivalent to the definition of derivation of  $f$

$(\forall \varepsilon > 0) (\exists k_0 > 0) (\forall k \geq k_0) :$

$$\left| \frac{|f(x) - f(\tilde{x})|}{|x - \tilde{x}|} - |f'(x)| \right| \leq \frac{\varepsilon}{2}.$$

It is easy to see that the following statement holds.

$(\forall n \in \mathbb{N}) (\forall \varepsilon > 0) (\exists k_n > 0) (\forall k \geq k_n) :$

$$\left| \frac{|f^n(x) - f^n(\tilde{x})|}{|x - \tilde{x}|} - \prod_{i=0}^{n-1} |f'(f^i x)| \right| \leq \frac{\varepsilon}{2}.$$

It is equivalent to

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{|f^n(x) - f^n(\tilde{x})|}{|x - \tilde{x}|} &= \prod_{i=0}^{n-1} |f'(f^i(x))| \\
\lim_{k \rightarrow \infty} \frac{1}{n} \log \frac{|f^n(x) - f^n(\tilde{x})|}{|x - \tilde{x}|} &= \frac{1}{n} \log \prod_{i=0}^{n-1} |f'(f^i(x))| \\
\lim_{k \rightarrow \infty} \left( \frac{1}{n} \log |f^n(x) - f^n(\tilde{x})| + \frac{k}{n} \right) &= \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^i(x))| \\
\left| \left( \frac{1}{n} \log |f^n(x) - f^n(\tilde{x})| + \frac{k}{n} \right) - \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^i(x))| \right| &\leq \frac{\varepsilon}{2}
\end{aligned} \tag{3.1}$$

From the Birkhoff Ergodic Theorem (Theorem 2.22) we have

$$L(f) = \int_a^b \log |f'| \, d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^i(x))|$$

which is equivalent to the following inequality

$(\forall \varepsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall n \geq n_0) :$

$$\left| L(f) - \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^i(x))| \right| \leq \frac{\varepsilon}{2}. \tag{3.2}$$

From equations 3.1 and 3.2 we get

$(\forall \varepsilon > 0) (\exists n_0 \in \mathbb{N}, k_0 > 0) (\forall n \geq n_0, k \geq k_0) :$

$$\left| L(f) - \left( \frac{1}{n} \log |f^n(x) - f^n(\tilde{x})| + \frac{k}{n} \right) \right| \leq \varepsilon.$$

And from definition of  $H_{n,k}$ :

$$|L(f) - H_{n,k}(x)| \leq \varepsilon.$$

■

### Example 3.13

In Figure 2 we show computed values of  $H_{n,k}$  of the Tent map for  $k = 300$ . As  $n$  grows we get closer to the Lyapunov exponent. Since we are on closed interval,  $n$  should not be too large, since the divergence can not grow to infinity.

### Example 3.14

The Lyapunov exponent of the Logistic map  $F_4 = 4x(1-x)$  is  $\log 2$  as we shew in Example 3.7. In Figure 2 we show the convergence of  $H_{n,k}$  to the Lyapunov exponent.

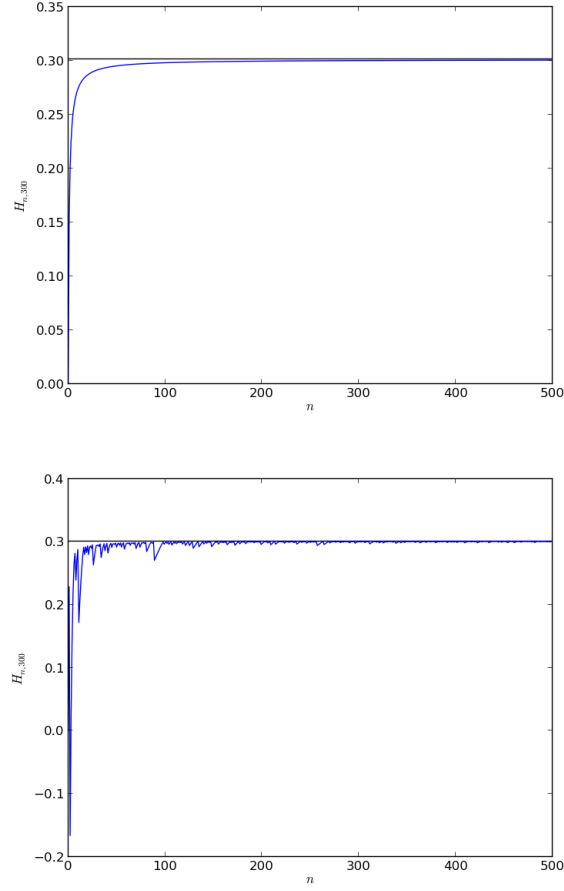


Figure 2: Convergence of  $H_{n,300}$  for Tent and Logistic maps ( $n = 1, \dots, 500$ )

### 3.1 Logistic family

We already know the Lyapunov exponent of the Logistic map

$$F_4(x) = 4x(1-x),$$

where  $x \in [0, 1]$ ,  $L(F_4(x)) = \log 2$ , see section Preliminaries.

This map is the special case of the parametric family, given by equation below

$$F_\mu = \mu x(1-x),$$

where  $x \in [0, 1]$  and  $\mu \in [0, 4]$  is the parameter.

In this section we study the behavior of the Logistic map in dependence on its parameter and then show how do the changes of parameter  $\mu$  affect the Lyapunov exponent. Firstly we define some terms we will use for describing the behavior of  $F_\mu$ .

Before we set the precise definition of bifurcation let us explain the intuitive understanding of it. Imagine the situation where the population of rabbits is hunted by foxes. If the population growth is bigger than number of rabbits killed by foxes, the population will grow to infinity. If the growth is equal to the number of eaten rabbits, the population will be in stagnation. And if foxes kill more rabbits than are born, the population will die out. This situation is very simplified and in nature would never happen but it is nice example for explaining the bifurcations. We see that there are three different possibilities. The overbreeding, the stagnation and the dying out. Each of them represents different dynamical behavior of the system. The value at which the behavior is changing is called *bifurcation boundary*. Another examples of bifurcations can be found in [29].

**Definition 3.15** [15, page 60] *Let  $f_\mu(x)$  be a parametrized family of functions. Then there is a bifurcation at  $\mu_0$  if there exists  $\epsilon > 0$  such that whenever  $\mu_1$  and  $\mu_2$  satisfy  $\mu_0 - \epsilon < \mu_1 < \mu_0$  and  $\mu_0 < \mu_2 < \mu_0 + \epsilon$ , then the dynamics of  $f_{\mu_1}(x)$  are different from the dynamics of  $f_{\mu_2}(x)$ . In other words, the dynamics of the function changes when the parameter value crosses through the point  $\mu_0$ .*

**Remark 3.16** Let  $f_\mu$  be a map dependent on the real parameter  $\mu$ . If there are two values of the parameter  $\mu_1, \mu_2$  (suppose  $\mu_1 < \mu_2$ ) such that  $f_{\mu_1}$  and  $f_{\mu_2}$  are not topologically equivalent then there exists a bifurcation boundary at some  $\mu_b$  such that  $\mu_1 \leq \mu_b \leq \mu_2$ .

For parameter between zero and one the map  $F_\mu$  has one attracting fixed point  $x_1 = 0$ . When  $\mu = 1$  the bifurcation occurs, the fixed point loses its stability and another attracting fixed point is born at  $x_2 = \frac{\mu-1}{\mu}$ .

When  $\mu$  passes three there comes the *period doubling*. That does mean that there is a sequence of bifurcation boundaries  $b_n$  such that if  $\mu$  passes  $b_{k+1}$  the periodic points of period  $k$  turn to be unstable and the new stable cycle of period  $2k$  is born. Mitchell Feigenbaum has proved in [10] that for sequence  $\{d_k\}_{k=1}^\infty$  where  $d_k$  is defined by

$$d_k = b_{k+1} - b_k$$

the following limit exists

$$\delta = \lim_{k \rightarrow \infty} \frac{d_k}{d_{k+1}}.$$

Such defined  $\delta$  is called the *Feigenbaum constant*. That does mean that there is a value of parameter where the period doubling ends. The approximate values of Feigenbaum constant and the end point of period doubling for Logistic family are

$$\delta \approx 4.669202, \quad b_F \approx 3.569945.$$

To study the behavior of the Logistic family for the parameter between the end of the period doubling and four, we should restrict the phase space to its *core* which is defined by

$$X_c = \left[ F^2 \left( \frac{1}{2} \right), F \left( \frac{1}{2} \right) \right].$$

The core is strongly invariant under  $F_\mu$ . Moreover all the points of  $X$  are mapped into  $X_c$  under some iteration. More precisely, for every point  $x$  of the phase space there exists  $n_0 \in \mathbb{N}$  such that for every  $n > n_0$  the point  $F_\mu^n(x)$  is in the core.

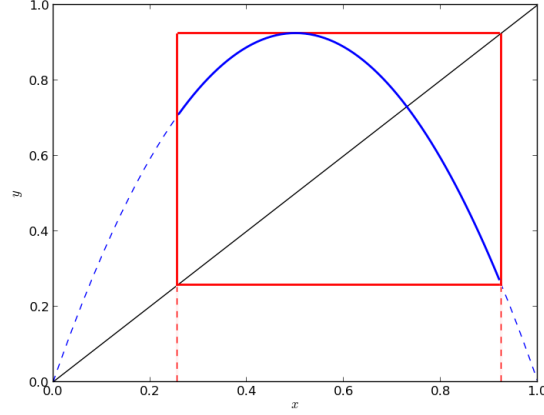


Figure 3: The core of  $F_{3.7}$ .

**Example 3.17**

The Logistic family for  $\mu \in [b_F, 4]$  restricted to the core is chaotic in sense of Definition 2.11 (*Devaney's definition of chaos*).

In Figure 4 is shown how the Lyapunov exponent changes in dependence on the behavior of the system. We can see that in bifurcation points the Lyapunov exponent is zero. Moreover the Lyapunov exponent is non-positive when  $\mu < b_F$ .



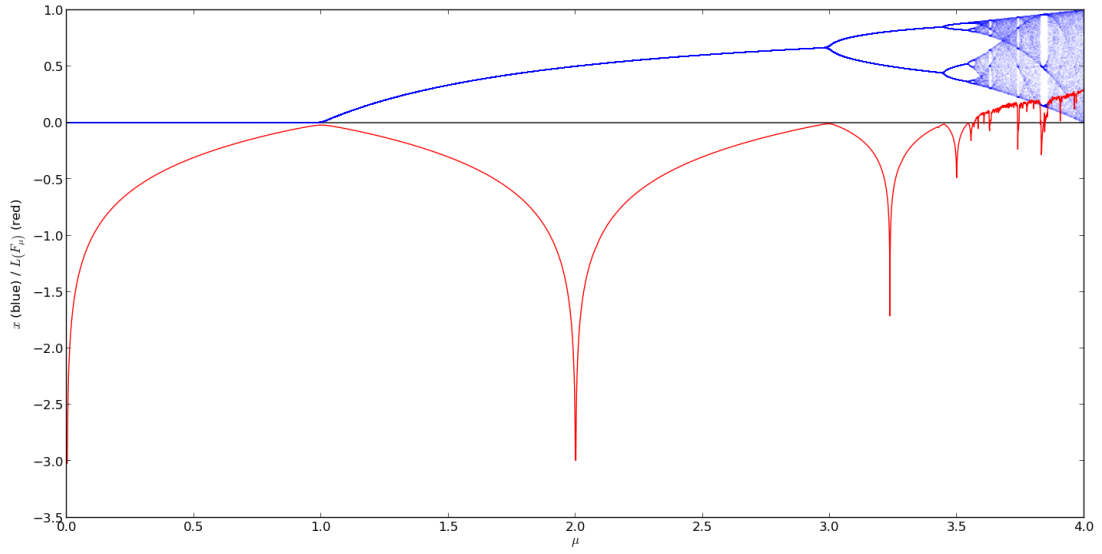


Figure 4: Bifurcation diagram (blue) and Lyapunov exponent of  $F_\mu$  (red).

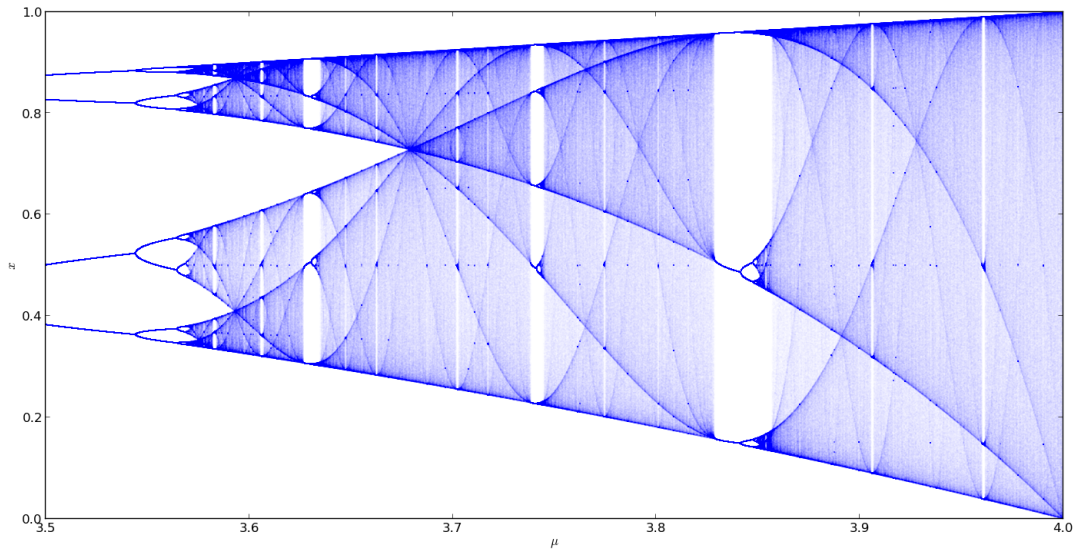


Figure 5: Bifurcation diagram of  $F_\mu$  for  $\mu \in [3.5, 4]$ .

## 4 Lyapunov exponents for $n$ -dimensional maps

Before we give the definition of Lyapunov exponents in higher dimensions let us show their geometrical interpretation. As in dimension one, the Lyapunov exponents measure the speed at which the distance of two neighboring points grows. The difference is that in multidimensional case we measure this speed in all the orthogonal directions. Imagine of the  $n$ -dimensional sphere should be constructed. This sphere is transformed under the map  $f$  onto  $n$ -dimensional ellipsoid. Each of the Lyapunov exponents correspond to one of the orthogonal semi-axes of that ellipsoid.

To set the definition we need the following theorems and definitions.

**Theorem 4.1 [6, page 311]** *Let  $X$  be a compact differentiable manifold with  $\dim X = m$ . (To simplify the notation we regard  $X$  as a subset of  $\mathbb{R}^m$ .) Let  $f : X \rightarrow X$  be a diffeomorphism and let  $\mu$  be a  $f$ -invariant ergodic probability measure on  $X$ . Then at almost every  $x \in X$  there exist a decomposition of  $\mathbb{R}^m$  into*

$$\mathbb{R}^m = E_1(x) \oplus \cdots \oplus E_r(x)$$

and constants

$$L_1 < \cdots < L_r$$

such that

(i) the limits

$$L_i = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D(f^n)(x) \mathbf{v}\| = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D(f^{-n})(x) \mathbf{v}\|$$

exist for  $0 \neq \mathbf{v} \in E_i(x)$ ,  $1 \leq i \leq r$ , and do not depend on  $\mathbf{v}$  or  $x$ ,

(ii)  $\dim E_i(x)$  is constant,  $1 \leq i \leq r$ , and

(iii)  $Df(x) E_i(x) = E_i(f(x))$

**Proof.** Take  $A(x) = Df(x)$  and apply the Theorem 4.3. ■

In the following definition and theorem we assume that  $X$  is a probability space,  $f : X \rightarrow X$  is a measure preserving transformation and  $A$  is  $m \times m$  invertible matrix. For  $n \geq 1$  we write

$$A_n(x) = A(f^{n-1}(x)) \cdots A(x).$$

If  $f$  is invertible, we write

$$A_{-n}(x) = A(f^{-n}(x))^{-1} \cdots A(f^{-1}(x))^{-1}.$$

**Definition 4.2** For  $x \in X$  and  $\mathbf{v} \in \mathbb{R}^m$ , define

$$\begin{aligned}\bar{L}_+(x, \mathbf{v}) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|A_n(x) \mathbf{v}\|, \\ \underline{L}_+(x, \mathbf{v}) &= \liminf_{n \rightarrow \infty} \frac{1}{n} \log \|A_n(x) \mathbf{v}\|.\end{aligned}$$

If  $\bar{L}_+ = \underline{L}_+$ , then we simply write  $L_+$ . Define  $\bar{L}_-$ ,  $\underline{L}_-$  and  $L_-$  similarly using  $A_{-n}(x)$ .

**Theorem 4.3** [6, page 307] Let  $f$  be an invertible measure preserving transformation on a probability space  $X$ . Then for almost every  $x \in X$  there exist numbers

$$L_1(x) < \cdots < L_{r(x)}(x)$$

and a decomposition of  $\mathbb{R}^m$  into

$$\mathbb{R}^m = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$$

such that

- (i)  $L_+(x, \mathbf{v}) = -L_-(x, \mathbf{v}) = L_i(x)$  for  $0 \neq \mathbf{v} \in E_i(x)$ ,
- (ii)  $A(x)E_i(x) = E_i(f(x))$ ,
- (iii)  $r(x)$ ,  $L_i(x)$  and  $\dim E_i(x)$  are constant along orbits of  $f$ .

**Proof.** Proof of Theorem 4.3 is in [6, page 308]. ■

**Definition 4.4** Let  $X$  be a compact differentiable manifold with  $\dim X = m$ . (To simplify the notation we regard  $X$  as a subset of  $\mathbb{R}^m$ .) Let  $f : X \rightarrow X$  be a diffeomorphism and let  $\mu$  be a  $f$ -invariant ergodic probability measure on  $X$ . The numbers

$$L_1, \dots, L_r$$

in Theorem 4.1 are called Lyapunov exponents.

In numerical experiments we use the following formula to compute Lyapunov exponents

$$L_i = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\sigma_i\|,$$

where  $\sigma_i$  are singular values of matrix  $D(f^n)$ . It flows from the next equation

$$\sigma_i = \|A\mathbf{v}_i\|,$$

where  $\mathbf{v}_i$  is an eigenvector of matrix  $A^T A$ . In our geometrical representation images of orthogonal eigenvectors  $\mathbf{v}_i$  are the orthogonal semi-axes of an ellipsoid. The length of these semi-axes are  $\sigma_i$  (for more details see [6]).

Unfortunately, Propositions 3.2 and 3.3 are not easy to extend for multi dimensional maps, those results are not known to the author. Hence these are open questions that are assumed with true answer, as mathematical folklore, and are standardly used for characterization of rational or irrational patterns. Hence:

- Open question 4.5** (i) If a continuous map on  $n$ -dimensional cube ( $I^n$  where  $n > 0$ ) has positive Lyapunov exponent, does it imply chaos? If it does, in which sense?
- (ii) If a continuous map on  $n$ -dimensional cube ( $I^n$  where  $n > 0$ ) has one or more of its Lyapunov exponents equal to zero, does it imply that there is a bifurcation boundary?
- (iii) If a continuous map on  $n$ -dimensional cube ( $I^n$  where  $n > 0$ ) has negative all of its Lyapunov exponents, does it imply Lyapunov stability?

However the question whether the positive Lyapunov exponents implies chaos or at least Lyapunov instability is not answered, the mathematicians suppose that if the maximal Lyapunov exponent is positive, the system is sensitively dependent on initial conditions. If it is negative, the system is supposed to be periodic or asymptotically periodic.

#### 4.1 Hénon map

This map was introduced by Michel Hénon in 1976 and it is one of the simplest nonlinear multidimensional maps. This discrete  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  map is given by equations below.

$$\begin{aligned}x_{n+1} &= y_n + 1 - ax_n \\ y_{n+1} &= bx_n,\end{aligned}$$

where  $a, b \in \mathbb{R}$  are parameters.

It is known that for  $a = 1.4$  and  $b = 0.3$  the Hénon map is chaotic. Let us use the method described in previous section to approximate the average Lyapunov exponents. To compute the approximations of Lyapunov exponents we take the logarithm of singular values of  $\prod_{i=1}^n J(x_i, y_i)$  divided by length of the orbit.  $J(x, y)$  is the Jacobian matrix of Hénon map. In our numerical experiments we computed the average over 100 orbits of length 200.

$$\begin{aligned}L_1 &\approx 0.183559 \\ L_2 &\approx -0.706437\end{aligned}$$

Since the bifurcation diagram of Hénon map has dimension 4, it can be plotted only when one of the parameters is fixed. In Figures 7 – 10 are the bifurcation diagrams of Hénon map with four different values of parameter  $a$ .

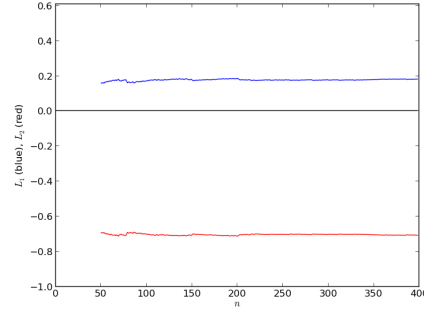


Figure 6: Convergence of  $\log \sigma_n/n$  for Hénon map with  $a = 1.4$ ,  $b = 0.3$ .

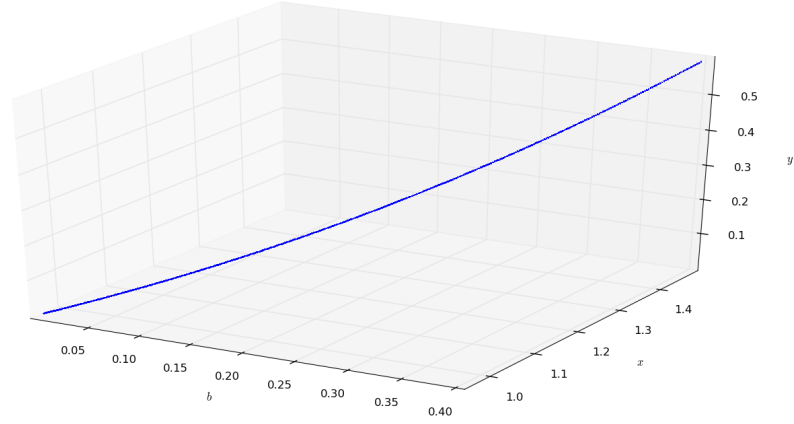


Figure 7: Bifurcation diagram of Hénon map for  $a = 0.05$

In Figure 11 are shown the Lyapunov exponents corresponding to bifurcation diagrams in Figures 7 – 10. Note that for  $a = 0.05$ ,  $a = 0.5$  and  $a = 0.95$  the maximal Lyapunov exponent is non-positive and it is equal to zero in bifurcation boundaries. See that for  $a = 1.4$ , where the system is chaotic, is the value of maximal Lyapunov exponent positive. This would be in agreement with the Proposition 3.2 if it was extendable to multidimensional case.

The Lyapunov exponents of Hénon map for parameters  $a \in [0, 1.4]$  and  $b \in [0, 0.3]$  are in Figures 12 and 13.

In Figure 14 is plotted the sign of the maximal Lyapunov exponent  $L_1$  of Hénon map. Blue color represents negative Lyapunov exponent and red color represents positive Lyapunov exponent.

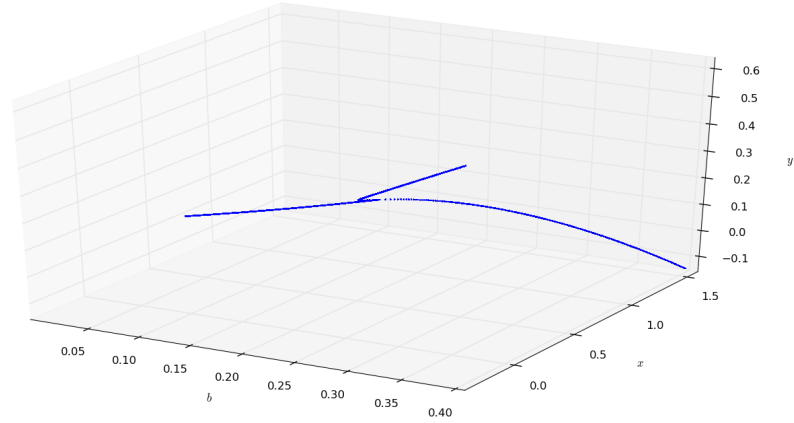


Figure 8: Bifurcation diagram of Hénon map for  $a = 0.5$

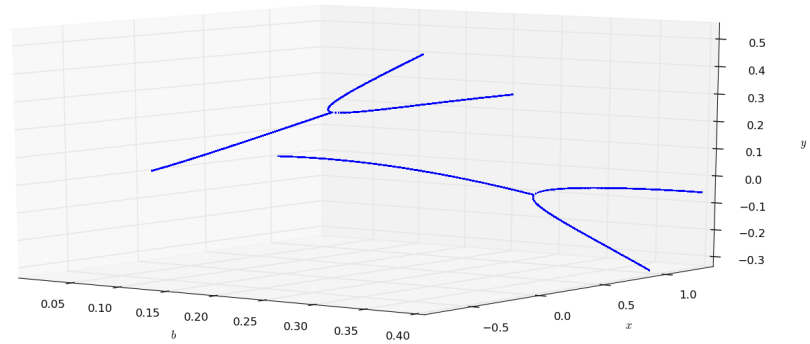


Figure 9: Bifurcation diagram of Hénon map for  $a = 0.95$

punov exponent. Here it is easy to see for which choice of parameters the system is showing chaotic motion, and for which one is the system exhibiting periodic or quasi-periodic movement. E.g. the chaotic situation is on Figure 11d for values of parameters  $a = 1.4, b = 0.3$  and the other is on Figure 11a for values of parameters  $a = 0.05, b = 0.15$ .

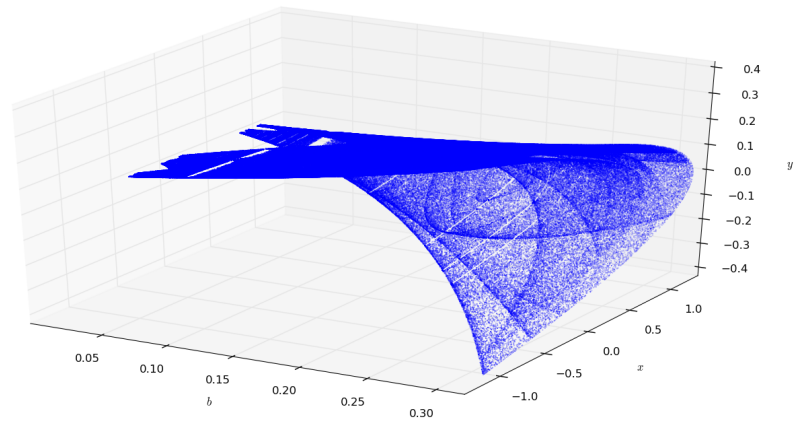
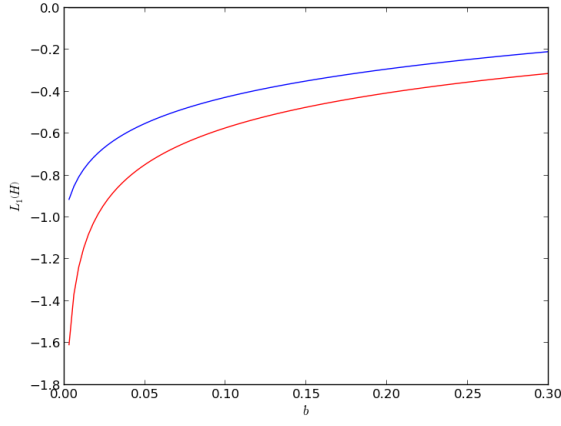
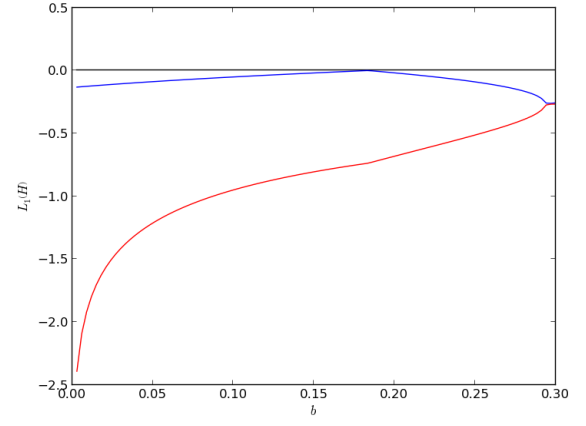
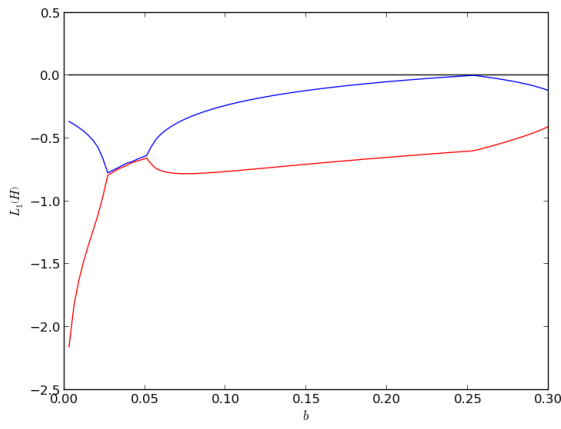
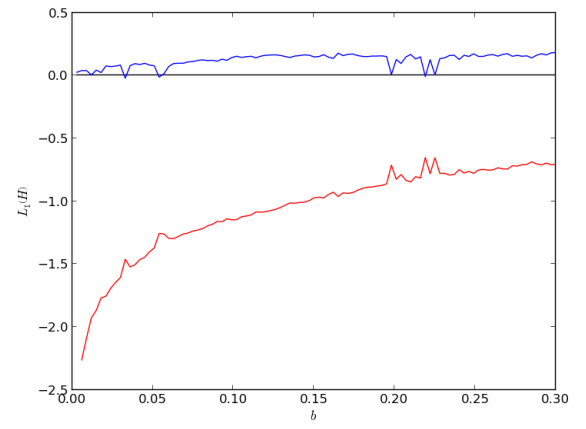


Figure 10: Bifurcation diagram of Hénon map for  $a = 1.4$

(a)  $a = 0.05$ (b)  $a = 0.5$ (c)  $a = 0.95$ (d)  $a = 1.4$ Figure 11: Lyapunov exponents of Hénon map for fixed  $a$ .



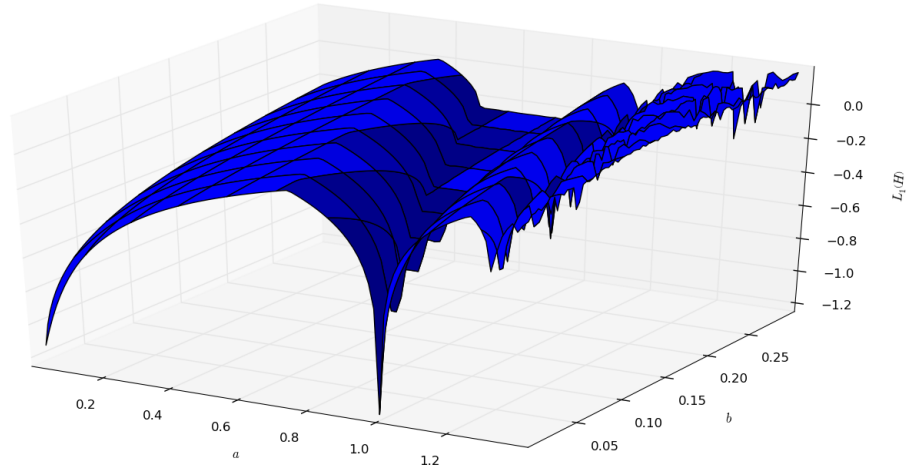


Figure 12: Lyapunov exponent  $L_1$  of Hénon map.

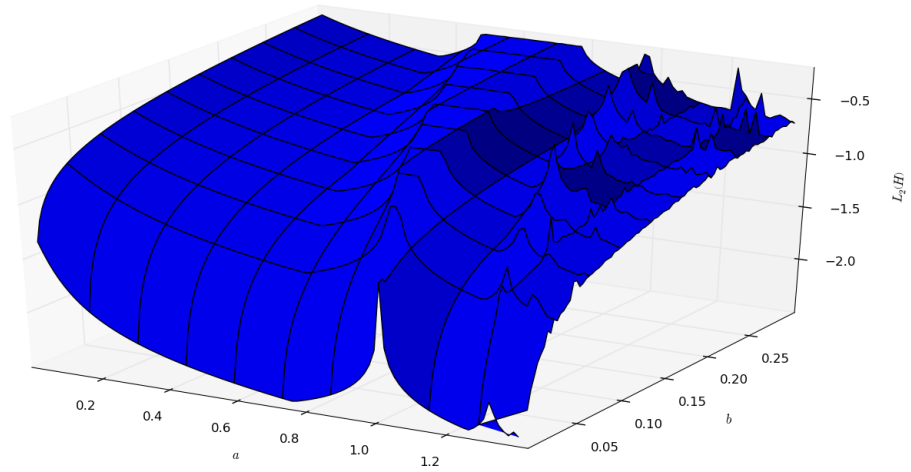


Figure 13: Lyapunov exponent  $L_2$  of Hénon map.

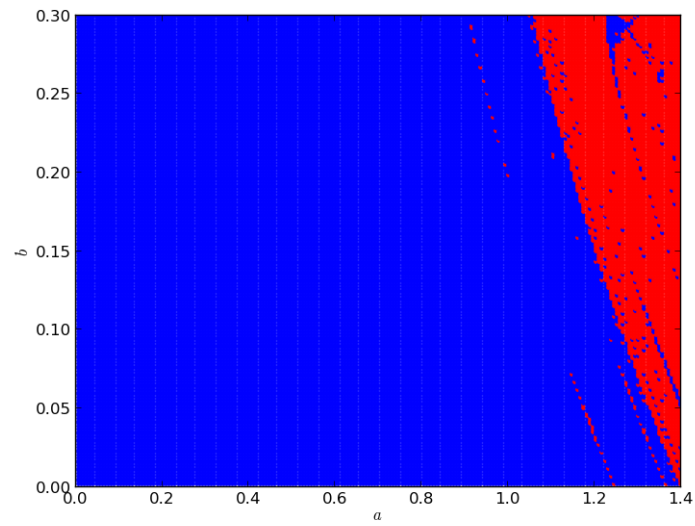


Figure 14: Sign of the maximal Lyapunov exponent (red = positive, blue = negative).

## 5 Lyapunov exponents of a Lotka–Volterra systems

Lotka–Volterra system describes populations of two species, the predator and the prey. The equations were set after the First World War by Vito Volterra. He was trying to describe the populations of some fish species in the Adriatic sea.

It is clear that the growth of predator population is positively influenced by the number of prey, hence the predation coefficient  $\delta$  is positive. But the growth rate of predators (parameter  $\gamma$ ) should be negative since it would die out without feeding. We assume that prey fish have infinite resources of food so the growth rate (parameter  $\alpha$ ) should be positive but the population decreases as predator fish kill the prey fish, hence the predation coefficient  $\beta$  is negative. The original differential equations are:

$$\begin{aligned}\dot{x} &= x(\alpha - \beta y), \\ \dot{y} &= y(-\gamma + \delta x)\end{aligned}$$

where  $\alpha, \beta, \gamma, \delta$  are positive,  $x$  denotes the number of prey and  $y$  is the number of predator.

This simple model doesn't describe the reality well. If there were no predators the prey fish population would grow to infinity. It is caused by the assumption of infinite resources of food but in reality this never happens and the fish of the same species have to fight for food. To model this situation the coefficients of interspecific competition  $\epsilon$  and  $\zeta$  are added. After this modification the equations would be:

$$\begin{aligned}\dot{x} &= x(\alpha - \epsilon x - \beta y), \\ \dot{y} &= y(-\gamma - \zeta y + \delta x)\end{aligned}$$

where the coefficients  $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$  are positive.

It is often useful to study the discrete model instead of continuous one in population modelling. The discrete dynamical system corresponding to the original Lotka–Volterra is given by difference equations below

$$\begin{aligned}x_{n+1} &= x_n(\alpha - \beta y_n), \\ y_{n+1} &= y_n(-\gamma + \delta x_n).\end{aligned}$$

Analogical competitive one is:

$$\begin{aligned}x_{n+1} &= x_n(\alpha - \epsilon x_n - \beta y_n), \\ y_{n+1} &= y_n(-\gamma - \zeta y_n + \delta x_n).\end{aligned}$$

In this section we compute the Lyapunov exponents for some special cases of discrete-time competitive Lotka–Volterra systems and study the dependence of these Lyapunov exponents on two real parameters. The values of Lyapunov exponents can indicate chaotic behavior of such systems or at least sensitive dependence on initial conditions.

One of the special cases of the competitive Lotka–Volterra is Sharkovskii's system given by

$$S(x, y) = (x(4 - x - y), xy),$$

where the phase space is a triangle with vertices  $(0, 0)$ ,  $(0, 4)$  and  $(4, 0)$ . The periodic properties of the restriction of  $S$  to the lower side of the triangle is clear since the restriction is equal to the Logistic map. The fixed points are  $\text{Fix}(S) = \{(0, 0), (1, 2), (3, 0)\}$ . The point  $(x, y) = (0, 0)$  is a saddle point, the others are repelling points. Let us now focus only on the interior of the invariant triangle. It was proved that there are no periodic points of periods 2 and 3 but for  $n \geq 4$  there is periodic point of period  $n$ . The cycle of period 4 was given explicitly in [2]. Moreover authors numerically approximated the cycle of period 5. In [25] was given explicitly the cycle of period 6. The question if the set of all the interior periodic points is dense is still open (it was stated as an open problem by A. N. Sharkovskii in [30]).

Now the task is to research all Lyapunov exponents of the Lotka–Volterra model with respect to relevant parameters. For this purpose we are going to use techniques introduced in the previous section. However, we are not able to use Theorem 4.1 directly since  $S$  is not a diffeomorphism. Fortunately it was proved in [34] that it is possible to split the phase space into regions such that  $S$  restricted to each of them is diffeomorphism.

Next, utilizing Source Code 3 one can compute Lyapunov exponents. Hence, the Lyapunov exponents of Sharkovskii's system are

$$\begin{aligned} L_1 &\approx 0.303881, \\ L_2 &\approx -0.009527. \end{aligned}$$

Consequently, the Sharkovskii's model has positive maximal Lyapunov exponent that yields chaotic pattern.

Let us denote the part  $4x - x^2$  by  $g(x)$ . It is easy to verify that  $g(x)$  is topologically conjugate to the Logistic map by conjugacy  $h(x) = 4x$ . Modifying  $g$  by parameter  $\mu \in [0, 1]$  we get the parametric family  $g(x) = \mu x(4 - x)$  that corresponds to the parametric system of Logistic map  $F_\mu(x)$  discussed in the section *Logistic family*. Moreover we modify the second equation by parameter  $\xi \in [0, 1]$ .

Finally, the parametrized model that should be researched is:

$$S_{\mu, \xi}(x, y) = (\mu x(4 - x) - xy, \xi xy).$$

In Figures 15-18 are the bifurcation diagrams of  $S_{\mu, \xi}$  for fixed parameter  $\mu$ . Combined with the Figure 19 they show us the change of behavior from periodic or quasi-periodic to chaotic. Again see that in first two cases (parameter values  $\mu = 0.25$  and  $\mu = 0.5$ ) the maximal Lyapunov exponent is non-positive (for  $\mu = 0.25$  it seems to be zero but it is lower than zero all the time) and it is equal to zero only in bifurcation boundaries. In the last two images we see that some values of the maximal Lyapunov exponents are positive which is considered as the indication of chaotic behavior of the system.

**Open problem 5.1** As simulations show there are dependences on initial values of tested points for Lyapunov exponents. The problem is more complex, a domain (phase space) should be detected. Hence:

*How to compute (simulate) Lyapunov exponents for models with non-trivial phase spaces?*

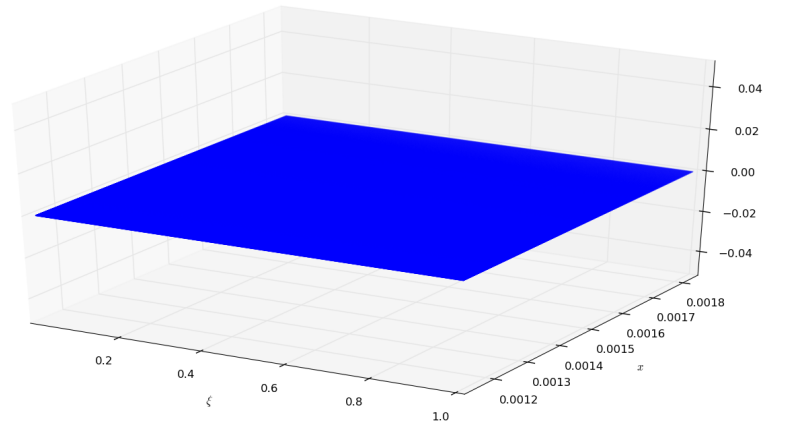


Figure 15: Bifurcation diagram of Lotka–Volterra family for  $\mu = 0.25$

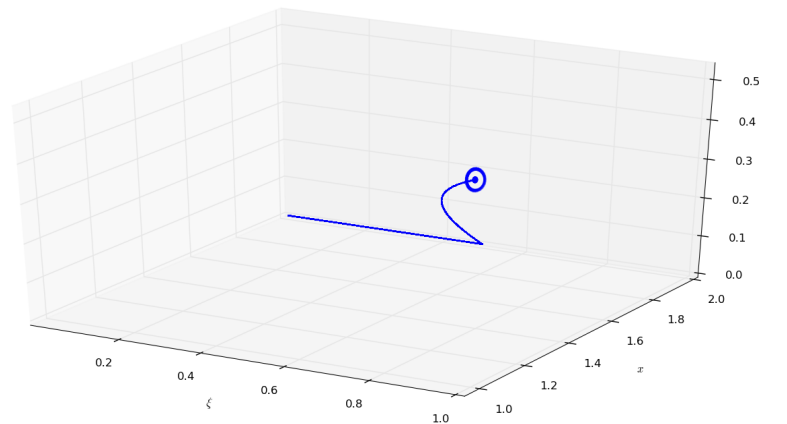


Figure 16: Bifurcation diagram of Lotka–Volterra family for  $\mu = 0.5$

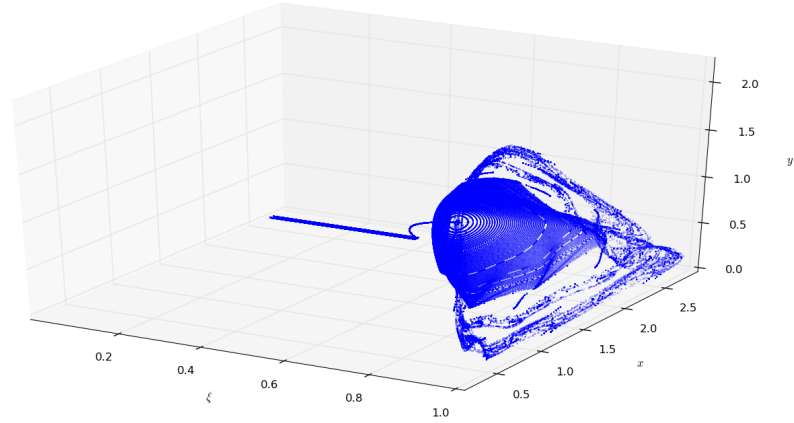


Figure 17: Bifurcation diagram of Lotka–Volterra family for  $\mu = 0.75$

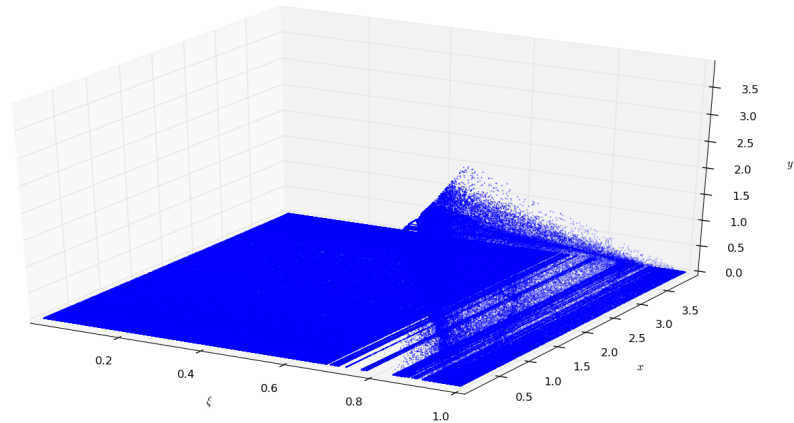


Figure 18: Bifurcation diagram of Lotka–Volterra family for  $\mu = 1.0$

Lyapunov exponents for parameters  $\mu \in [0, 1]$  and  $\xi \in [0, 1]$  are in Figures 20 and 21. Let us note that for  $\xi = 0$  the maximal Lyapunov exponents of  $S_{\mu,\xi}$  correspond to the Lyapunov exponents of Logistic family  $F_\mu$  to which the restriction of  $S_{\mu,\xi}$  is conjugate.

In Figure 22 is plotted the sign of the maximal Lyapunov exponent  $L_1$ . Blue color represents negative Lyapunov exponents and red color represents the positive ones. Here it is easy to see for which choice of parameters the system is showing chaotic motion, and

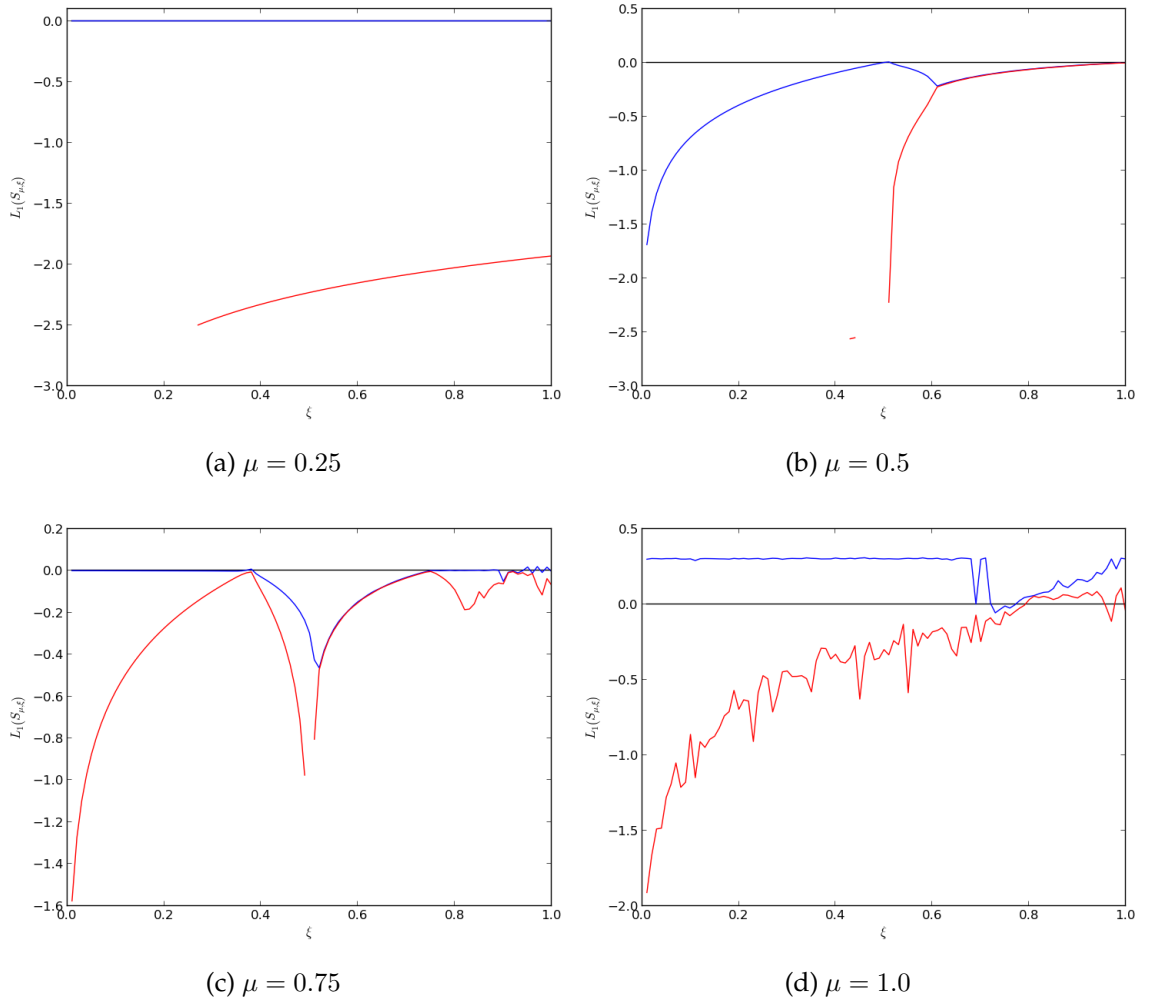


Figure 19: Lyapunov exponents of Lotka–Volterra family for fixed  $a$ .

for which one is the system exhibiting periodic or quasi-periodic movement. E.g. the chaotic situation is on Figure 19d for parameter values  $\mu = 1$ ,  $\xi = 1$  and the other is on the Figure 19b for values of parameters  $\mu = 0.5$ ,  $\xi = 0.2$ .

Notice the parametrized model, that is researched, has identical dynamical properties like  $F_4(x)$  for  $\xi = 0$ . That clearly follows from the fact that the second iteration equals to  $F_4(x)$ , namely

$$S_{\mu,0}(x, y) = (\mu x(4 - x) - xy, 0)$$

and

$$S_{\mu,0}^2(x, y) = (\mu\alpha(4 - \alpha) - \alpha \cdot 0, 0) = (\mu\alpha(4 - \alpha), 0)$$

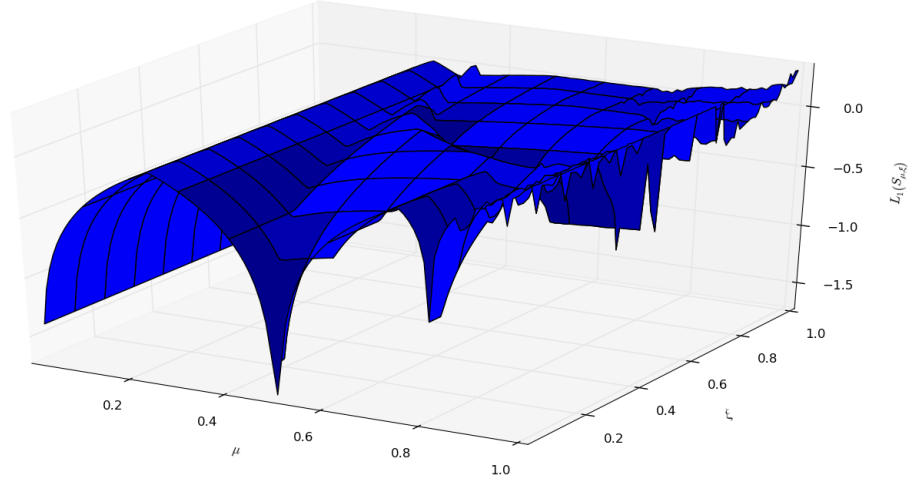


Figure 20: Lyapunov exponent  $L_1$  of Lotka–Volterra family.

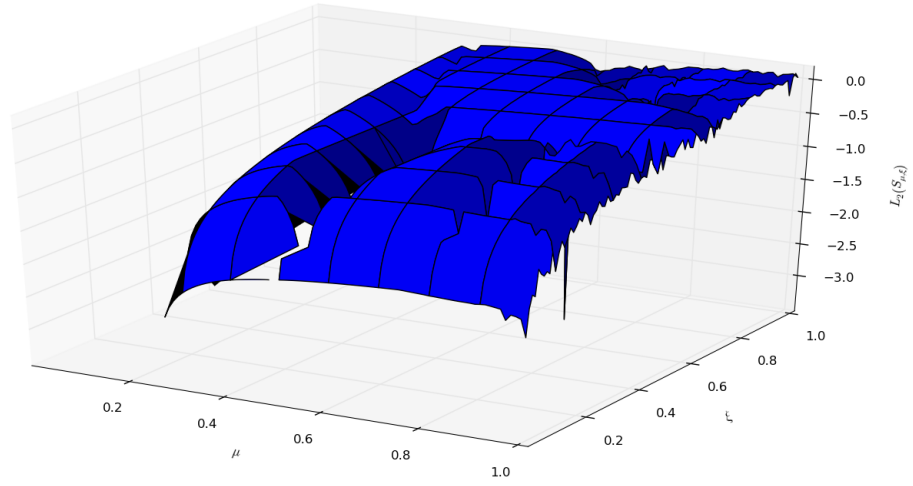


Figure 21: Lyapunov exponent  $L_2$  of Lotka–Volterra family.



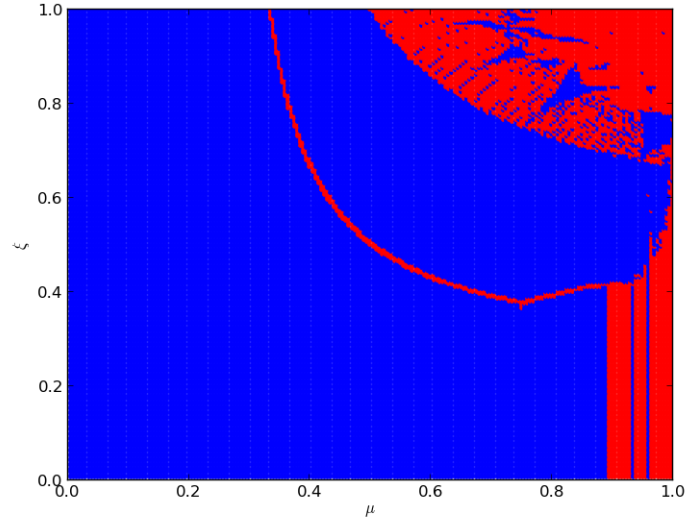


Figure 22: Sign of the maximal Lyapunov exponent (red = positive, blue = negative).

where  $\alpha = \mu x(4 - x) - xy$ , and

$$S_{\mu,0}^{n+1}(x,y) = (F_{\mu}^n(x), 0)$$

for any  $n > 0$  ending the observation.

The maximal Lyapunov exponent of this restriction is equal to the Lyapunov exponent of the Logistic map (compare Figure 20 and Figure 4). In Figure 22 can be seen that maximal Lyapunov exponents for  $\mu, \xi \in [0.9, 1] \times [0, 0.42]$  are mostly positive. That corresponds to positive Lyapunov exponent of the Logistic family behind the Feigenbaum point. This means that the parameter  $\xi \in [0, 0.42]$  doesn't significantly affect the system. As  $\xi$  passes 0.42 the behavior starts to be more complex, the second coordinate of  $S_{\mu,\xi}$  becomes significant.

## 6 Implementation

In this section we explain the algorithms which were used in numerical experiments and briefly introduce Python 2.7 in which all the numerical experiments were implemented.

Python 2.7 is interpreted scripting language which does mean that the source code doesn't have to be compiled and it is executed by an interpreter while it is running. The greatest advantage of Python 2.7 is that the source codes are short and easy to read. It is multi-platform and open-source language with many libraries for specific use such as SciPy and NumPy which are an alternative for some MATLAB functions. For the introduction to Python 2.7 see [11].

Here we give the overlook at libraries used for numerical experiments:

- decimal – library used for variable precision arithmetic and work with big numbers
- pylab – library for plotting
- math – library for basic mathematical operations
- numpy – library used for work with matrices, alternative to MATLAB functions
- scipy – library for scientific calculations, alternative to MATLAB functions
- sympy – library for symbolic calculations

### Algorithms for one-dimensional cases

The following code computes the value of  $H_{n,k}$  for the Logistic map directly from definition 3.11.

---

```

1  # import packages
2  from decimal import *
3  from math import *
4  import pylab
5
6  # set the precision
7  getcontext().prec = 400
8
9  # set the initial values
10 itermax = 500                # number of iterations
11 k = Decimal(300.0)          # k
12 x = [Decimal(pi) - Decimal(3.0)] # seed 1
13 xt = [x[0] + Decimal(10**(-k))] # seed 2
14
15 # set the initial value of  $H_{1,k}$ 
16 H = [Decimal(Decimal(log(abs(x[0] - xt[0]), 10)) / (Decimal(1.0) + (k / (Decimal(1.0)))))]
17
18 for i in range(itermax):
```

---

```

19
20     # create the orbit of seeds
21     x.append(Decimal(4)*x[i]*(Decimal(1)-x[i]))
22     xt.append(Decimal(4)*xt[i]*(Decimal(1)-xt[i]))
23
24     # compute the value of  $H_{n,k}$ 
25     H.append(Decimal(Decimal(log(abs(x[i+1] - xt[i+1]),10))/(Decimal(1.0*i+2)) + (k/(Decimal(1.0*i
        +2))))))
26
27     # plot the results
28     pylab.plot([log(2,10) for i in range(len(H))], 'k')
29     pylab.plot(H)
30     pylab.xlabel(r'$n$')
31     pylab.ylabel(r'$H_{n,300}$')
32     pylab.show()

```

---

Source Code 1: Algorithm for computing  $H_{n,300}$  for the Logistic map

The next algorithm draws the bifurcation diagram of Logistic map and computes its Lyapunov exponent. The bifurcation diagram is made by iterating the initial point and then plotting the last iterations. The Lyapunov exponent is computed by approximating the sum obtained from the alternative definition.

---

```

1  # import packages
2  import pylab
3  from math import *
4
5  # set initial values and number of iterations
6  itermac = 200      # number of iterations
7  finalits = 130     # first iterations which are not plotted
8  mu = [0.0]         # parameter mu
9  z=[None]           # initial value of Lyapunov exponent
10
11 # draw a black line at y = 0.0
12 pylab.plot([i/1000.0 for i in range(4001)], [0 for i in range(4001)], 'k')
13
14 # draw bifurcation diagram and Lyapunov exponent
15 for i in range(4000):
16
17     # set initial point
18     x = [0.4]
19
20     # append the next value of mu
21     mu.append((i+1)/1000.0)
22     for n in range(itermac):
23
24         # create the orbit of x
25         x.append(mu[i+1]*x[n]*(1-x[n]))
26
27     # get last (itermac-finalits) iterations and plot them
28     x_p = x[finalits:]
29     y = [mu[i+1] for item in x_p]
30     pylab.plot(y, x_p, 'b.', markersize=0.1)

```

---

```

31
32     # compute Lyapunov exponent at current mu
33     if abs(mu[i+1]*(1-2*item))==0:
34         z.append(None)
35     else:
36         pom = [log(abs(mu[i+1]*(1-2*item))),10) for item in x]
37         z.append(sum(pom)/(itermax*1.0))
38
39 #plot results
40 pylab.plot(mu,z,'r')
41 pylab.xlabel(r'$\mu$')
42 pylab.ylabel(r'$x_{(blue)} / L \left( F_{\mu} \right)_{(red)}$')
43 pylab.show()

```

---

Source Code 2: Algorithm for drawing bifurcation diagram and computing of Lyapunov exponent of Logistic map

### Algorithms for $n$ -dimensional cases

The next code shows the convergence of the sequence  $\log \sigma_i / n$  to the Lyapunov exponents of the Hénon map.

---

```

1  # import packages
2  from decimal import *
3  from math import *
4  import numpy as np
5  import pylab
6
7  # set the precision
8  getcontext().prec = 1000
9
10 #set the initial values
11 N = 400                                # length of the orbit
12 a = Decimal(1.4)                       # value of the parameter a
13 b = Decimal(0.3)                       # value of the parameter b
14 x = [[Decimal(0.0),Decimal(0.0)]]     # initial point
15 L1 = []                                # list of approximations of Lyapunov exponent L1
16 L2 = []                                # list of approximations of Lyapunov exponent L2
17
18 # get the initial point on the attractor
19 for i in range(2000):
20     x[0] = [x[0][1]+Decimal(1.0) - a*x[0][0]*x[0][0] , x[0][0]*b]
21
22 # create the orbit
23 for s in range(N):
24     x.append([x[s][1]+Decimal(1.0) - a*x[s][0]*x[s][0] , x[s][0]*b])
25
26 # the Jacobian matrix at initial value
27 A = np.array ([[Decimal(-2.0)*a*x[0][0] , Decimal(1.0)] , [b , Decimal(0.0)]], dtype=np.dtype(
    Decimal))
28
29 for n in range(N-1):

```

---

```

30
31     # compute the product of Jacobian matrices
32     A = np.dot(np.array ([[ Decimal(-2.0)*a*x[n+1][0] , Decimal(1.0)] , [b , Decimal(0.0)]], dtype=
        np.dtype(Decimal)),A)
33
34     # get the singular values from eigenvalues of A^T * A
35     B = np.dot(A, A.T)
36     lam1 = (B[0][0]+B[1][1] + Decimal((B[0][0]+B[1][1])*(B[0][0]+B[1][1]) - Decimal(4.0)*(B[0][0]*
        B[1][1] - B[0][1]*B[1][0]) ).sqrt() )/(Decimal(2.0))
37     lam2 = (B[0][0]+B[1][1] - Decimal((B[0][0]+B[1][1])*(B[0][0]+B[1][1]) - Decimal(4.0)*(B[0][0]*
        B[1][1] - B[0][1]*B[1][0]) ).sqrt() )/(Decimal(2.0))
38     lam1 = Decimal(lam1).sqrt()
39     lam2 = Decimal(lam2).sqrt()
40
41     # get the approximation of Lyapunov exponents
42     L1.append(log(lam1,10)/(1.0*n+1))
43     L2.append(log(lam2,10)/(1.0*n+1))
44
45     # remove first 50 iterations
46     for i in range(len(L1)):
47         if i<50:
48             L1[i]=None
49             L2[i]=None
50
51     # plot the results
52     pylab.ylim([-1.0,0.61])
53     pylab.plot([0 for i in range(len(L1))], 'k')
54     pylab.plot(L1, 'b')
55     pylab.plot(L2, 'r')
56     pylab.xlabel(r'$n$')
57     pylab.ylabel(r'$L_{1}$ (blue), $L_{2}$ (red)')
58     pylab.show()

```

---

Source Code 3: Convergence of  $\sigma_n/n$ 

This algorithm draws a bifurcation diagram for fixed parameter  $a$ . The orbit of the initial point is plotted except the first few iterations.

---

```

1  # import packages
2  from mpl_toolkits.mplot3d import Axes3D
3  from pylab import *
4
5  # set the initial values
6  itermax = 1500                # maximum iterations
7  finalits = 130                # first iterations which are not plotted
8  a = 0.4 #0.4 #0.9 #1.9       # parameter a (fixed)
9  b = [0.0]                    # initial value of parameter b
10 z=[None]
11
12 # create the figure in 3D
13 fig = figure()
14 ax = fig.add_subplot(111, projection='3d')
15

```

---

```

16 for i in range(750):
17     #set the initial point
18     x = [[0.0, 0.0]]
19
20
21     # append the next value of b
22     b.append((i+1)/1000.0)
23
24     # get the initial point on the attractor
25     for s in range(2000):
26         x[0] = [x[0][1]+1.0 - a*x[0][0]*x[0][0] , x[0][0]*b[i+1]]
27
28     # create the orbit of x
29     for n in range(itermax):
30         x.append([x[n][1]+1.0 - a*x[n][0]*x[n][0] , x[n][0]*b[i+1]])
31
32     # get last (itermac-finalits) iterations and plot them
33     x_p = x[ finalits :]
34     x = [item[0] for item in x_p]
35     y = [item[1] for item in x_p]
36     z = [b[i+1] for item in x_p]
37     ax.scatter(z, x, y, 'b.', s=0.1, edgecolor='b')
38
39 #plot results
40 ax.set_xlabel(r'$b$')
41 ax.set_ylabel(r'$x$')
42 ax.set_zlabel(r'$y$')
43 show()

```

---

Source Code 4: Drawing bifurcation diagram of Hénon map for fixed  $a$

In the following code the Lyapunov exponents of the Hénon map are computed for fixed parameter  $a$ . For every value of  $b$  the Lyapunov exponents are approximated by  $\log \sigma_i/n$ . The algorithm is constructed to compute the average of the values over more orbits. The variable *sampleSize* determines the number of used orbits.

---

```

1  # import packages
2  from decimal import *
3  from math import *
4  import numpy as np
5  from pylab import *
6
7  # set precision and Emax
8  getcontext().prec = 1000
9  getcontext().Emax = 10**10000
10
11 # method which computes the Lyapunov exponents for given parameters [L1,L2] = lyap(par1,par2)
12 def lyap(par1,par2):
13
14     # set initial values
15     N = 200                                # length of the orbit
16     a = Decimal(par1)                       # parameter a
17     b = Decimal(par2)                       # parameter b

```

---

```

18     x = [[ Decimal(0.0),Decimal(0.0)]]    # seed
19     L1 = []                                # list of approximations of Lyapunov exponent L1
20     L2 = []                                # list of approximations of Lyapunov exponent L2
21     sampleSize = 1                          # number of samples
22
23     # get the initial point on the attractor
24     for i in range(2000):
25         x[0] = [x[0][1]+Decimal(1.0) - a*x[0][0]*x[0][0] , x[0][0]*b]
26
27     # create the orbits
28     for s in range(N*sampleSize):
29         x.append([x[s][1]+Decimal(1.0) - a*x[s][0]*x[s][0] , x[s][0]*b])
30
31     # compute Lyapunov exponents
32     for s in range(sampleSize):
33
34         # the Jacobian matrix at initial point
35         A = np.array ([[ Decimal(-2.0)*a*x[s*N][0] , Decimal(1.0)] , [b , Decimal(0.0)]] , dtype=np.
                        dtype(Decimal))
36
37         for n in range(N-1):
38
39             # compute the product of Jacobian matrices along orbits
40             A = np.dot(np.array ([[ Decimal(-2.0)*a*x[s*N+n+1][0] , Decimal(1.0)] , [b , Decimal
                        (0.0)]] , dtype=np.dtype(Decimal)),A)
41
42
43         # get the singular values from eigenvalues of A * A^T
44         A = np.dot(A, A.T)
45         lam1 = (A[0][0]+A[1][1] + Decimal((A[0][0]+A[1][1])*(A[0][0]+A[1][1]) - Decimal(4.0)*(A
                        [0][0]*A[1][1] - A[0][1]*A[1][0]) ).sqrt() )/(Decimal(2.0))
46         lam2 = (A[0][0]+A[1][1] - Decimal((A[0][0]+A[1][1])*(A[0][0]+A[1][1]) - Decimal(4.0)*(A
                        [0][0]*A[1][1] - A[0][1]*A[1][0]) ).sqrt() )/(Decimal(2.0))
47         lam1 = Decimal(lam1).sqrt()
48         lam2 = Decimal(lam2).sqrt()
49
50         # get approximations of Lyapunov exponents
51         L1.append(eval(str(lam1.log10()/Decimal(1.0*N))))
52         L2.append(eval(str(lam2.log10()/Decimal(1.0*N))))
53
54     # compute and return the average Lyapunov exponents
55     ave_L1 = np.sum(L1)/(1.0*len(L1))
56     ave_L2 = np.sum(L2)/(1.0* len(L2))
57     return [ave_L1, ave_L2]
58
59     # set the initial values
60     a = 0.05 #0.5 #0.95 #1.4    # parameter a
61     par_b = []                  # parameter b
62     z_a=[]                      # list of approximations of Lyapunov exponent L1
63     z_b=[]                      # list of approximations of Lyapunov exponent L2
64
65     # compute the Lyapunov exponents for every (a,b)
66     for i in range(100):
67

```

---

```

68     par_b.append(3*(i+1)/1000.0)
69     z_a.append([])
70     z_b.append([])
71     pom = lyap(a, par_b[i])
72     z_a[i].append(pom[0])
73     z_b[i].append(pom[1])
74
75     # plot results
76     plot(par_b,[0 for i in par_b], 'k')
77     plot(par_b, z_a)
78     plot(par_b, z_b, 'r')
79     xlabel(r'$b$')
80     ylabel(r'$L_{1}$\ left (\_Hright)$')
81     show()

```

---

Source Code 5: Plotting the Lyapunov exponents of Hénon map for fixed  $a$

The next code is similar to the one above. The difference is that the value of parameter  $a$  is no longer fixed and the approximations are computed for all the pairs  $(a, b)$ .

---

```

1  # import packages
2  from decimal import *
3  from math import *
4  import numpy as np
5  from mpl_toolkits.mplot3d import Axes3D
6  from pylab import *
7
8  # create figures in 3D
9  fig1 = figure()
10 ax1 = fig1.add_subplot(111, projection='3d')
11 fig2 = figure()
12 ax2 = fig2.add_subplot(111, projection='3d')
13
14 # set precision and Emax
15 getcontext().prec = 1000
16 getcontext().Emax = 10**10000
17
18 # method which computes the Lyapunov exponents for given parameters [L1,L2] = lyap(par1,par2)
19 def lyap(par1,par2):
20
21     # set initial values
22     N = 200                                # length of the orbit
23     a = Decimal(par1)                      # parameter a
24     b = Decimal(par2)                      # parameter b
25     x = [[Decimal(0.0),Decimal(0.0)]]     # seed
26     L1 = []                               # list of approximations of Lyapunov exponent L1
27     L2 = []                               # list of approximations of Lyapunov exponent L2
28     sampleSize = 1                        # number of samples
29
30     # get the initial point on the attractor
31     for i in range(200):
32         x[0] = [x[0][1]+Decimal(1.0) - a*x[0][0]*x[0][0] , x[0][0]*b]
33

```



---

```

34     # create the orbits
35     for s in range(N*sampleSize):
36         x.append([x[s][1]+Decimal(1.0) - a*x[s][0]*x[s][0] , x[s][0]*b])
37
38     # compute Lyapunov exponents
39     for s in range(sampleSize):
40
41         # the Jacobian matrix at initial point
42         A = np.array ([[ Decimal(-2.0)*a*x[s*N][0] , Decimal(1.0)] , [b , Decimal(0.0)]], dtype=np.
            dtype(Decimal))
43
44         for n in range(N-1):
45
46             # compute the product of Jacobian matrices along orbits
47             A = np.dot(np.array ([[ Decimal(-2.0)*a*x[s*N+n+1][0] , Decimal(1.0)] , [b , Decimal
                (0.0)]], dtype=np.dtype(Decimal)),A)
48
49             # get the singular values from eigenvalues of A * A^T
50             A = np.dot(A, A.T)
51             lam1 = (A[0][0]+A[1][1] + Decimal((A[0][0]+A[1][1])*(A[0][0]+A[1][1]) - Decimal(4.0)*(A
                [0][0]*A[1][1] - A[0][1]*A[1][0]) ).sqrt() )/(Decimal(2.0))
52             lam2 = (A[0][0]+A[1][1] - Decimal((A[0][0]+A[1][1])*(A[0][0]+A[1][1]) - Decimal(4.0)*(A
                [0][0]*A[1][1] - A[0][1]*A[1][0]) ).sqrt() )/(Decimal(2.0))
53             lam1 = Decimal(lam1).sqrt()
54             lam2 = Decimal(lam2).sqrt()
55
56             # get approximations of Lyapunov exponents
57             L1.append(eval(str(lam1.log10()/Decimal(1.0*N))))
58             L2.append(eval(str(lam2.log10()/Decimal(1.0*N))))
59
60         # compute and return the average Lyapunov exponents
61         ave_L1 = np.sum(L1)/(1.0*len(L1))
62         ave_L2 = np.sum(L2)/(1.0* len(L2))
63         return [ave_L1, ave_L2]
64
65     # set the initial values
66     par_a = []      # parameter a
67     par_b = []      # parameter b
68     z_a=[]          # list of approximations of Lyapunov exponent L1
69     z_b=[]          # list of approximations of Lyapunov exponent L2
70
71     # create list of values of parameter b
72     for i in range(90):
73         par_b.append(3*(i+1)/900.0)
74
75     # compute the Lyapunov exponents for every (a,b)
76     for i in range(100):
77
78         # append a value of parameter a
79         par_a.append(14*(i+1)/1000.0)
80         z_a.append([])
81         z_b.append([])
82         for j in range(90):
83

```

---

```
84         # compute the Lyapunov exponents for each parametrization
85         pom = lyap(par_a[i], par_b[j])
86         z_a[i].append(pom[0])
87         z_b[i].append(pom[1])
88
89     # plot results
90     X, Y = np.meshgrid(par_a, par_b)
91     Z_a = np.array(z_a)
92     Z_b = np.array(z_b)
93     ax1.plot_surface(X, Y, Z_a.T)
94     ax2.plot_surface(X, Y, Z_b.T)
95     ax1.set_xlabel(r'$a$')
96     ax1.set_ylabel(r'$b$')
97     ax1.set_zlabel(r'$L_{1}$\ left (H\right)$')
98     ax2.set_xlabel(r'$a$')
99     ax2.set_ylabel(r'$b$')
100    ax2.set_zlabel(r'$L_{2}$\ left (H\right)$')
101    show()
```

---

Source Code 6: Plotting the Lyapunov exponents of Hénon map

## 7 Conclusions

The main aim of this thesis was to measure the chaos in discrete dynamical systems, namely the parametric family of Lotka–Volterra system. This was done by computing the Lyapunov exponents of that systems.

In the first section the most important ideas of the chaos theory were recalled. In the second section the brief introduction to discrete dynamical systems, ergodic theory and topological conjugacy was made. The basic definitions and theorems were set. Then the Lyapunov exponent was introduced and studied in one-dimensional case. The Lyapunov exponent of Logistic parametric family was observed since it is known for its chaotic behavior for some values of parameter (see Figure 4).

The multidimensional case was studied on Hénon map, which is known for its chaotic behavior for some values of parameters. The main results came in section *Lyapunov exponents of a Lotka–Volterra systems* where the parametric family of the Lotka–Volterra system was introduced and its Lyapunov exponents were computed (Figures 20, 21 and 22). The behavior of the system was observed on bifurcation diagrams and then compared with the maximal Lyapunov exponent (see Figures 15, 16, 17, 18 and 19). For numerical approximations and simulations were used algorithms from section *Implementation*. The implementation was done in Python 2.7 language.

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